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On the Behavior of the Fundamental Solution of the Heat Equation with Variable Coefficients*

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1. Introduction

The solution $q(t, x, y)$ of the equation

$$\frac{\partial q}{\partial t} = \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2 q}{\partial x_i^2} \tag{1.1}$$

with the boundary condition $q(t, x, y) \to \delta_x(y)$ as $t \to 0$ can of course be written explicitly as

$$q(t, x, y) = (2\pi t)^{-k/2} \exp\left\{-\frac{1}{2t}\|x - y\|^2\right\}, \tag{1.2}$$

where $\|x - y\|$ denotes the Euclidean distance. Looking at (1.2) one sees immediately that

$$\lim_{t\to 0} [-2t \log q(t, x, y)] = \|x - y\|^2 . \tag{1.3}$$

We shall consider the analogue of (1.1). Let $p(t, x, y)$ be the solution of the equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij}(x) \frac{\partial^2 p}{\partial x_i \, \partial x_j} \tag{1.4}$$

with the boundary condition $p(t, x, y) \to \delta_x(y)$ as $t \to 0$. We prove a formula similar to (1.3):

$$\lim_{t\to 0} [-2t \log p(t, x, y)] = d^2(x, y) , \tag{1.5}$$

where $d(x, y)$ is the distance induced by a Riemannian metric derived from the coefficients $\{a_{ij}(x)\}$.

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If G is a region with a boundary B, we consider the solution $\phi(x, \lambda)$ of the equation

$$\tfrac{1}{2} \sum a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \, \partial x_j} = \lambda \phi \qquad \text{for} \qquad x \in G$$

with the boundary value $\phi = 1$ on B. We prove that

$$\lim_{\lambda \to \infty} [-(2\lambda)^{-1/2} \log \phi(x, \lambda)] = d(x, B) \,, \tag{1.6}$$

where x is any point of G and $d(x, B)$ is the shortest distance to the boundary B from x.

The behavior (1.5) of $p(t, x, y)$ is derived from the behavior (1.6) of $\phi(x, \lambda)$ which is proved first. Although these results are closely related to certain properties of Markov processes, the probabilistic connections will be explored separately, [3].

Section 2 covers the preliminaries. The assumptions and the main theorems are stated there. The later sections cover the actual proof.

2. Preliminaries

L stands for the following differential operator acting on smooth functions on R_k:

$$Lf = \tfrac{1}{2} \sum a_{ij}(x) \frac{\partial^2 f}{\partial x_i \, \partial x_j} \,. \tag{2.1}$$

The matrix $\{a(x)\}$ of coefficients is of course symmetric and positive definite and the following assumptions are made concerning $a(x)$:

(i) Uniform Hölder condition, i.e., for all i and j we have

$$|a_{ij}(x) - a_{ij}(y)| \leqq M \, \|x - y\|^\alpha \text{ for all } x, y, \qquad \alpha > 0.$$

(2.2) (ii) Uniform ellipticity condition, i.e., there exist constants α and A such that for any vector $\xi_1, \cdots, \xi_k$, and x

$$A \sum \xi_i^2 \geqq \sum a_{ij}(x) \xi_i \xi_j \geqq \alpha \sum \xi_i^2 \,.$$

We then consider the equation

$$\frac{\partial p}{\partial t} = Lp \tag{2.3}$$

with the boundary condition $p(t, x, y) \to \delta_x(y)$ as $t \to 0$.

The following theorem concerning the existence of a solution $p(t, x, y)$ for (2.3) can be found in [1], [2].

THEOREM 2.1. *Under the hypothesis* (2.2) *on the coefficients there exists a solution* $p(t, x, y)$ *for equation* (2.3) *and it has the following properties:*

(i) $p(t, x, y) \geqq 0$,

(ii) $\int p(t, x, y) p(s, y, z) \, dy = p(t + s, x, z)$,

(iii) $p(t, x, y)$ *is continuous in* t, x, *and* y,

(iv) *there are constants* M *and* α *such that*

$$p(t, x, y) \leqq M t^{-k/2} \exp\left\{-\frac{\alpha}{2t} \|x - y\|^2\right\},$$

(v) *there are positive constants* α_1, α_2, M_1, M_2, λ *such that*

$$p(t, x, y) \geqq M_1 t^{-k/2} \exp\left\{-\frac{\alpha_1}{2t} \|x - y\|^2\right\}$$

$$- M_2 t^{-k/2+\lambda} \exp\left\{-\frac{\alpha_2}{2t} \|x - y\|^2\right\}.$$

Further $p(t, x, y)$ *is unique.*

We now introduce the metric d. Let $p(\tau)$, $0 \leqq \tau \leqq 1$, be a smooth path in R_k. Then the length of such a path is defined as

$$l(p) = \int_0^1 [\dot{p}(\tau) a^{-1}(p(\tau)) \dot{p}(\tau)]^{1/2} \, d\tau \,,$$

where $\dot{p}(\tau)$ stands for $dp(\tau)/d\tau$, $a^{-1}(p(\tau))$ for the matrix inverse to $a(p(\tau))$ and $(\theta a \theta)$ for the quadratic form $\Sigma a_{ij} \theta_i \theta_j$; $l(p)$ is the natural length in a metric defined locally as

$$ds^2 = \sum a^{ij}(x) \, dx_i \, dx_j \,.$$

The operator L modified by a first order term is Laplacian in this metric. The global distance $d(x, y)$ induced by this metric is defined as

$$d(x, y) = \inf_{p: \, \substack{p(0)=x \\ p(1)=y}} l(p) \,. \tag{2.4}$$

We can now state the first main theorem concerning the behavior of $p(t, x, y)$ introduced in Theorem 2.1 for small t.

THEOREM 2.2.

$$\lim_{t \to 0} [-2t \log p(t, x, y)] = d^2(x, y)$$

uniformly over x *and* y *such that* $d(x, y)$ *is bounded.*

We consider an open set G in R_k. Let B be the boundary of G. It is assumed that every point b on B is a limit point of the exterior of G.

Let $\phi(x, \lambda)$ be the solution of the equation

$$\text{(2.5)}\qquad \begin{aligned} L\phi &= \lambda\phi \qquad \text{for} \qquad x \in G, \\ \phi &= 1 \qquad \text{for} \qquad b \in B. \end{aligned}$$

The solution exists for $\lambda \geqq 0$, if the boundary value is properly interpreted. The next theorem is concerned with the behavior of the solution $\phi(x, \lambda)$ for large λ.

THEOREM 2.3. *With the assumptions stated above,*

$$\lim_{\lambda\to\infty}\left[-\frac{1}{\sqrt{2\lambda}}\log\phi(x,\lambda)\right] = d(x, B)$$

uniformly over compact subsets of $G \cup B$.

We state some notation and a few elementary consequences concerning the distance $d(x,y)$. The distance $[(x-y)\Lambda(x-y)]^{1/2}$ is denoted by $\|x-y\|_\Lambda$, where Λ is a positive definite matrix. If Λ is chosen as $a^{-1}(z)$, then $\|x-y\|$ is denoted by $\|x-y\|_z$; $\|x-y\|$ is of course the Euclidean distance. It is clear from the assumptions that d, $\|\ \|$, $\|\ \|_z$ are all equivalent in the sense that the ratio of any two is bounded by a universal constant. We shall denote by $\Delta(x,y)$ any one of these metrics. If a and b are positive definite matrices, we shall mean by $a \geqq b$ that $a - b$ is positive semidefinite. The following is a consequence of the Hölder condition in (2.2).

LEMMA 2.4. *There exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that if $\Delta(x,y) \leqq \varepsilon$, then*

$$[1-\delta(\varepsilon)]a(x) \leqq a(y) \leqq [1+\delta(\varepsilon)]a(x),$$

$$[1-\delta(\varepsilon)]a^{-1}(x) \leqq a^{-1}(y) \leqq [1+\delta(\varepsilon)]a^{-1}(x).$$

From Lemma 2.4 and the definition of $d(x,y)$ we deduce

LEMMA 2.5. *There exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that if $\Delta(x,y) \leqq \varepsilon$ and $\Delta(x, z) \leqq \varepsilon$, then*

$$[1-\delta(\varepsilon)]\,\|x-y\|_z \leqq d(x,y) \leqq [1+\delta(\varepsilon)]\,\|x-y\|_z.$$

3. Elliptic Equation

In this section we prove Theorem 2.3. The method consists of comparing our solution with the solution of an equation having constant coefficients in a small domain and then piecing together the estimate with the help of the maximum principle.

Let y be an arbitrary point in R_k. Let the set S_ε and its boundary B_ε be defined as follows:

$$S_\varepsilon = [x: \|x-y\|_y < \varepsilon],$$

$$B_\varepsilon = [x: \|x-y\|_y = \varepsilon].$$

We denote by L_y the operator

$$L_y f = \tfrac{1}{2}\sum a_{ij}(y)\frac{\partial^2 f}{\partial x_i\,\partial x_j}.$$

It is an operator with constant coefficients. Let $\psi_\varepsilon(x, \lambda)$ be the solution to the equation

$$\text{(3.1)}\qquad \begin{aligned} L_y\psi_\varepsilon &= \lambda\psi_\varepsilon \qquad \text{for} \qquad x \in S_\varepsilon, \\ \psi_\varepsilon &= 1 \qquad \text{for} \qquad x \in B_\varepsilon. \end{aligned}$$

THEOREM 3.1. *For every $\lambda > 0$, $\psi(x, \lambda)$ is explicitly given by the formula below for $x \in S_\varepsilon$:*

$$\psi_\varepsilon(x,\lambda) = \frac{\cosh(\sqrt{2\lambda}\,r)}{\cosh(\sqrt{2\lambda}\,\varepsilon)} \qquad \textit{for} \qquad k = 1,$$

$$\psi_\varepsilon(x,\lambda) = \frac{\int_0^\pi \cosh(\sqrt{2\lambda}\,r\cos\theta)\sin^{k-2}\theta\,d\theta}{\int_0^\pi \cosh(\sqrt{2\lambda}\,\varepsilon\cos\theta)\sin^{k-2}\theta\,d\theta} \qquad \textit{for} \qquad k \geqq 2,$$

where $r = \|x-y\|_y$. For all k, $\psi_\varepsilon(x, \lambda)$ is a convex function of x.

Proof: The proof is elementary and consists of direct verification.

THEOREM 3.2. *For every $\rho > 0$ there exists a constant $M_\rho < \infty$ depending only on the dimension k such that, for all y, ε and λ,*

$$\psi_\varepsilon(y,\lambda) \leqq M_\rho \exp\{-\varepsilon(1-\rho)\sqrt{2\lambda}\}.$$

Proof: When $x = y$ or $r = 0$ the numerator in Theorem 3.1 reduces to a dimension constant. As for the denominator, it can be estimated from below by limiting the range of integration to the region $0 \leqq \theta \leqq \cos^{-1}(1-\rho)$.

We now consider the sets

$$T_\varepsilon = [x: d(x,y) < \varepsilon],$$

$$D_\varepsilon = [x: d(x,y) = \varepsilon].$$

Then in view of Lemma 2.5, there exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that the set

$$S = [x: \|x - y\|_y < \varepsilon(1 - \delta(\varepsilon))]$$

is contained in T_ε. Let B be the boundary of S:

$$B = [x: \|x - y\|_y = \varepsilon(1 - \delta(\varepsilon))] .$$

Obviously B is contained in $T_\varepsilon \cup D_\varepsilon$.

Let $f_\varepsilon(x, \lambda)$, $g_\varepsilon(x, \lambda)$, and $h_\varepsilon(x, \lambda)$ denote the solutions of the following equations:

$$\text{(3.2)} \quad \begin{aligned} Lf &= \lambda f && \text{for} \quad x \in T_\varepsilon , \\ f &= 1 && \text{for} \quad x \in D_\varepsilon , \end{aligned}$$

$$\text{(3.3)} \quad \begin{aligned} Lg &= \lambda g && \text{for} \quad x \in S , \\ g &= 1 && \text{for} \quad x \in B , \end{aligned}$$

$$\text{(3.4)} \quad \begin{aligned} L_y h &= \lambda h && \text{for} \quad x \in S , \\ h &= 1 && \text{for} \quad x \in B . \end{aligned}$$

LEMMA 3.3.

$$f(y, \lambda) \leqq g(y, \lambda) .$$

Proof: From equation (3.2) it is obvious that $f \leqq 1$ in $T_\varepsilon \cup D_\varepsilon$. Since $B \subset T_\varepsilon \cup D_\varepsilon$, it follows that $f \leqq 1$ on B. But f and g satisfy the same equation in S and on the boundary B of S, $g = 1$ and $f \leqq 1$. Hence $f \leqq g$ everywhere in S. Therefore in particular, $f(y, \lambda) \leqq g(y, \lambda)$.

LEMMA 3.4. *There exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that*

$$g(y, \lambda) \leqq h(y, \lambda(1 - \delta(\varepsilon))) .$$

Proof: By Theorem 3.1, h is convex and by Lemma 2.4

$$a(x) \leqq \frac{1}{1 - \delta(\varepsilon)} a(y) .$$

Therefore,

$$\tfrac{1}{2} \sum a_{ij}(x) \frac{\partial^2 h}{\partial x_i \, \partial x_j} \leqq \frac{1}{2(1 - \delta(\varepsilon))} \sum a_{ij}(y) \frac{\partial^2 h}{\partial x_i \, \partial x_j}$$

$$= \frac{1}{(1 - \delta(\varepsilon))} L_y h$$

$$= \frac{\lambda}{(1 - \delta(\varepsilon))} h ,$$

or

$$Lh(x, \lambda(1 - \delta(\varepsilon))) \leqq \lambda h \quad \text{in} \quad S .$$

But

$$Lg(x, \lambda) = \lambda g \quad \text{in} \quad S ,$$

and h and g have the same boundary value on B. Hence the solution $g(x, \lambda)$ is smaller than $h(x, \lambda(1 - \delta(\varepsilon)))$ in S. In particular,

$$g(y, \lambda) \leqq h(y, \lambda(1 - \delta(\varepsilon))) .$$

THEOREM 3.5. *The solution $f_\varepsilon(x, \lambda)$ of equation (3.2) has the following property. There exist a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a constant $M_\rho < \infty$ for every $\rho > 0$ such that, for all ε, y and λ,*

$$f_\varepsilon(y, \lambda) \leqq M_\rho \exp\{-(1 - \delta(\varepsilon))(1 - \rho)\varepsilon\sqrt{2\lambda}\} .$$

The proof is immediate from Theorem 3.2 and Lemmas 3.3 and 3.4.

Let G be any arbitrary region with a boundary B. Let $\phi(x, \lambda)$ be the solution to the equation

$$\text{(3.5)} \quad \begin{aligned} L\phi &= \lambda\phi && \text{for} \quad x \in G , \\ \phi &= 1 && \text{for} \quad x \in B . \end{aligned}$$

If G is unbounded, we determine ϕ by the additional restriction that $0 \leqq \phi \leqq 1$. If G is bounded, it is automatically true that $0 \leqq \phi \leqq 1$. Let

$$\text{(3.6)} \quad d(x, B) = \inf_{b \in B} d(x, b) .$$

We then have the following estimate concerning $\phi(x, \lambda)$.

THEOREM 3.6. *For every $\rho > 0$, there exists a function $C_\rho(x)$ such that*

$$\phi(x, \lambda) \leqq C_\rho(x) \exp\{-(1 - \rho) d(x, B)\sqrt{2\lambda}\} .$$

$C_\rho(x)$ depends only on $d(x, B)$ and is uniformly bounded whenever x varies in such a way that $d(x, B)$ remains bounded.

Proof: Let $x \in G$ be arbitrary. Then the set

$$S_R = [y: d(x, y) < d(x, B)]$$

is obviously contained in G and its boundary is

$$B_R = [y: d(x, y) = d(x, B)] .$$

Hence if $\psi(x, \lambda)$ is the solution of

$$\text{(3.7)} \qquad \begin{aligned} L\psi &= \lambda\psi \quad \text{for} \quad y \in S_R , \\ \psi &= 1 \quad \text{for} \quad y \in B_R , \end{aligned}$$

then

$$\phi(x, \lambda) \leqq \psi(x, \lambda) .$$

To estimate $\psi(x, \lambda)$ we choose ε small so that $n\varepsilon = d(x, B)$ and define the sets $S_1, S_2, \cdots S_n$ as follows:

$$S_j = [y: d(x,y) < j\varepsilon] ,$$

$$B_j = [y: d(x,y) = j\varepsilon] .$$

Let ψ_j be the solution to the equation

$$\text{(3.8)} \qquad \begin{aligned} L\psi_j &= \lambda\psi_j \quad \text{for} \quad y \quad \text{in} \quad S_j , \\ \psi_j &= 1 \quad \text{for} \quad y \quad \text{on} \quad B_j . \end{aligned}$$

We estimate ψ_{j+1} on B_j. It is obvious that if $b_j \in B_j$, the set $S = [y: d(b_j, y) < \varepsilon]$ is contained in S_{j+1} and hence

$$\psi_{j+1}(b_j, \lambda) \leqq f_\varepsilon(b_j, \lambda) ,$$

where f_ε refers to the function defined in equation (3.2). Therefore,

$$\text{(3.9)} \qquad \psi_{j+1}(b_j, \lambda) \leqq M_\rho \exp\{-[1 - \delta(\varepsilon)](1 - \rho)\sqrt{2\lambda}\,\varepsilon\} .$$

Now we can use the inequality

$$\phi(b_j, \lambda) \leqq \psi_{j+1}(b_j, \lambda) \sup_{b_{j+1} \in B_{j+1}} \phi(b_{j+1}, \lambda)$$

which with (3.9) leads to

$$\text{(3.10)} \qquad \phi(x, \lambda) \leqq [M_\rho]^n \exp\{-[1 - \delta(\varepsilon)](1 - \rho)n\varepsilon\sqrt{2\lambda}\} .$$

Since $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and since $n\varepsilon = d(x, B)$, this leads immediately to the theorem.

For the rest of the section we shall be concerned with obtaining a lower bound.

Let y be an arbitrary point in R_k, let S be the set

$$S = [x: \|x - y\|_y < \varepsilon]$$

and B its boundary

$$B = [x: \|x - y\|_y = \varepsilon] .$$

Γ is that part of the boundary B defined by

$$\Gamma = [x: \|x - y\|_y = \varepsilon , \|x - z\|_y < \varepsilon\sqrt{2\eta}] ,$$

where z is a point of B and η is a positive number. So Γ is a small portion of the boundary around z.

Let $\phi(x, \lambda)$ be the solution to the equation

$$\text{(3.11)} \qquad \begin{aligned} L\phi &= \lambda\phi \quad \text{for} \quad x \quad \text{in} \quad S , \\ \phi &= 1 \quad \text{for} \quad x \quad \text{on} \quad \Gamma , \\ \phi &= 0 \quad \text{for} \quad x \quad \text{on} \quad B - \Gamma . \end{aligned}$$

We have the following theorem giving us a lower bound on $\phi(y, \lambda)$.

THEOREM 3.7. *There exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a constant M_ρ for every $\rho > 0$ such that for all y and λ*

$$\phi(y, \lambda) \geqq \exp\{-\varepsilon(1 + \delta(\varepsilon))\sqrt{2\lambda}\}[1 - M_\rho \exp\{-\varepsilon(\eta - \rho - \delta(\varepsilon))\sqrt{2\lambda}\}] .$$

Proof: Let θ be a vector in R_k. For any z in R_k, θz is the scalar product $\Sigma\, \theta_i z_i$. For x in S we have

$$\begin{aligned} Le^{\theta(x-y)} &= \tfrac{1}{2}[\theta a(x)\theta]e^{\theta(x-y)} \\ &\geqq \tfrac{1}{2}(1 - \delta(\varepsilon))(\theta\Lambda\theta)e^{\theta(x-y)} , \end{aligned}$$

where $\Lambda = a(y)$. Putting $\lambda = \frac{1}{2}(1 - \delta(\varepsilon))[\theta\Lambda\theta]$, we have

$$Le^{\theta(x-y)} \geqq \lambda e^{\theta(x-y)} \quad \text{for} \quad x \quad \text{in} \quad S .$$

Hence if $g(x, \lambda)$ is the solution to

$$\text{(3.12)} \qquad \begin{aligned} Lg &= \lambda g \quad \text{in} \quad S , \\ g &= e^{\theta(x-y)} \quad \text{for} \quad x \quad \text{on} \quad B , \end{aligned}$$

then

$$\text{(3.13)} \qquad e^{\theta(x-y)} \leqq g(x, \lambda) .$$

Let $\psi(x, \lambda)$ be the solution to

$$\text{(3.14)} \qquad \begin{aligned} L\psi &= \lambda\psi \quad \text{for} \quad x \quad \text{in} \quad S , \\ \psi &= 1 \quad \text{for} \quad x \quad \text{on} \quad B . \end{aligned}$$

Then from Theorem 3.2, there is, for every $\rho > 0$, a constant M_ρ such that

$$\text{(3.15)} \qquad \psi(y, \lambda) \leqq M_\rho \exp\{-\varepsilon(1 - \rho)\sqrt{2\lambda}\} .$$

Let us now take the vector $\theta = k\Lambda^{-1}(z - y)$, where the constant k is to be chosen later, such that

(3.16) $$\lambda = \tfrac{1}{2}(1 - \delta(\varepsilon))[\theta\Lambda\theta] .$$

The following estimates are easily obtained:

$$\sup_{x\in B} \theta(x - y) = k\varepsilon^2 ,$$

$$\sup_{x\in B-\Gamma} \theta(x - y) = k\varepsilon^2(1 - \eta) .$$

Hence on B,

(3.17) $$e^{\theta(x-y)} \leqq e^{k\varepsilon^2(1-\eta)+k\varepsilon^2}\chi_\Gamma(x) ,$$

where $\chi_\Gamma(x)$ is the indicator function of Γ. Let us recall that $\phi(x, \lambda)$ satisfies the equation

(3.18) $$L\phi = \lambda\phi \quad \text{for } x \text{ in } S , \qquad \phi = \chi_\Gamma(x) \quad \text{for } x \text{ on } B ,$$

and compare the equations (3.12), (3.14) and (3.18). The functions g, ψ, and ϕ satisfy the same equation in S but with different boundary values $e^{\theta(x-y)}$, 1, and $\chi_\Gamma(x)$, respectively, on B. Since (3.17) is valid on the boundary, the following inequality is valid throughout S:

(3.19) $$g(x, \lambda) \leqq e^{k\varepsilon^2(1-\eta)}\psi(x, \lambda) + e^{k\varepsilon^2}\phi(x, \lambda) .$$

Setting $x = y$ in (3.19) and using (3.13), we obtain

$$e^{k\varepsilon^2(1-\eta)}\psi(y, \lambda) + e^{k\varepsilon^2}\phi(y, \lambda) \geqq 1 .$$

Using now the estimate (3.15) for $\psi(y, \lambda)$, we get

(3.20) $$\phi(y, \lambda) \geqq e^{-k\varepsilon^2} - e^{-k\varepsilon^2}\eta M_\rho e^{-\varepsilon(1-\rho)\sqrt{2\lambda}} .$$

Now (3.16) leads to the value $\varepsilon k = [1 + \delta(\varepsilon)]\sqrt{2\lambda}$. Substitution of this value of k in (3.20), a little simplification and replacement of $\delta(\varepsilon)$ by a larger function which again tends to zero as $\varepsilon \to 0$ yields

$$\phi(y, \lambda) \geqq e^{-\varepsilon(1+\delta(\varepsilon))\sqrt{2\lambda}}(1 - M_\rho e^{-\varepsilon(\eta-\rho-\delta(\varepsilon))\sqrt{2\lambda}}) .$$

This completes the proof.

Let y and z be two arbitrary points in R_k with $d(x, y) = \varepsilon$. Let S be the set

$$S = [x: d(x, z) \leqq 2\varepsilon\sqrt{\eta}] .$$

We consider the solution $\phi(x, \lambda)$ of the equation

(3.21) $$L\phi = \lambda\phi \quad \text{for } x \text{ in the exterior of } R_k - S , \qquad \phi = 1 \quad \text{for } x \text{ on the boundary } B \text{ of } S .$$

We further assume that η is small enough so that y is in the exterior of S. Then we have

THEOREM 3.8. *There exists a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, and a constant M_ρ for every $\rho > 0$ such that, for all y and z,*

$$\phi(y, \lambda) \geqq \exp\{-\varepsilon(1 + \delta(\varepsilon))\sqrt{2\lambda}\}[1 - M_\rho \exp\{-(\eta - \rho - \delta(\varepsilon))\sqrt{2\lambda}\}] .$$

Proof: Let $\varepsilon' = \|y - z\|_y$. Choose η' such that the set

$$E = [x: \|x - z\|_y \leqq \varepsilon'\sqrt{2\eta'}]$$

is contained in S. Let F denote the boundary

$$F = [x: \|x - z\|_y = \varepsilon'\sqrt{2\eta'}]$$

of S. Let $\psi(x, \lambda)$ be the solution to the equation

(3.22) $$L\psi = \lambda\psi \quad \text{for } x \text{ in the exterior of } E , \qquad \psi = 1 \quad \text{for } x \text{ on } F .$$

Let S' be the set

$$S' = [x: \|x - y\|_y < \varepsilon']$$

with boundary

$$B' = [x: \|x - y\|_y = \varepsilon']$$

and Γ' the set $\Gamma' = B' \cap E$. Let $g(x, \lambda)$ be the solution of

(3.23) $$Lg = \lambda g \quad \text{for } x \text{ in } S' , \qquad g = \chi_{\Gamma'}(x) \quad \text{for } x \text{ on } B' .$$

Since $E \subset S$ and ψ is the solution to (3.22) in the exterior of E, ψ is defined on the boundary B of S and $0 \leqq \psi \leqq 1$ on B. But ϕ is the solution of (3.21) in the exterior of S and therefore for x in the exterior of S

(3.24) $$\phi(x, \lambda) \geqq \psi(x, \lambda) .$$

Let A be the set $S' \cap$ (exterior of E). Its boundary is given by $[B' \cap$ (exterior of E)$] \cup (F \cap S')$. Then g and ψ are both solutions of the same equation $Lf = \lambda f$ in A. As for their boundary values,

$$\text{on } B' \cap (\text{exterior of } E), \quad g = 0 , \quad \psi \geqq 0 ,$$

$$\text{on } F \cap S', \quad g \leqq 1 , \quad \psi = 1 .$$

Hence throughout A, $\psi \geqq g$. Since y is in A, we have with (3.24)

$$\phi(y, \lambda) \geqq g(y, \lambda) .$$

Using Theorem 3.7 to estimate $g(y, \lambda)$, we have

$$g(y, \lambda) \geqq \exp\{-\varepsilon'(1 + \delta(\varepsilon'))\sqrt{2\lambda}\}[1 - M_\rho \exp\{-(\eta' - \rho - \delta(\varepsilon'))\sqrt{2\lambda}\}] .$$

It is easy to estimate using Lemma 2.5:

$$\varepsilon' \leqq \varepsilon(1 + \delta(\varepsilon))$$

and

$$\eta' \geqq 2\eta\left(\frac{1 - \delta(\varepsilon)}{1 + \delta(\varepsilon)}\right)^2 .$$

Therefore for ε small enough, $\eta' \geqq \eta$ and this proves the theorem.

Let y, z be two points with $d(y, z) = d$. Let n be an integer such that $n\varepsilon = d$. Let G be the sphere around z of radius $4\varepsilon\sqrt{\eta}$:

$$G = [x: d(x, z) \leqq 4\varepsilon\sqrt{\eta}] .$$

Let $\phi(x, \lambda)$ be the solution of the equation

$$\text{(3.25)} \qquad \begin{aligned} L\phi &= \lambda\phi \quad \text{for } x \text{ in the exterior of } G , \\ \phi &= 1 \quad \text{on the boundary of } G . \end{aligned}$$

THEOREM 3.9. *For the solution ϕ of equation* (3.25) *there exist a function $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a constant M_ρ for every $\rho > 0$ such that*

$$\phi(y, \lambda) \geqq \exp\{-d(1 + 4\sqrt{\eta})(1 + \delta(\varepsilon))\sqrt{2\lambda}\}$$
$$\times [1 - M_\rho \exp\{-\varepsilon(1 - 4\sqrt{\eta})(\eta - \rho - \delta(\varepsilon))\sqrt{2\lambda}\}]^n .$$

Proof: Let the points $y = y_0, y_1, y_2, \cdots, y_n = z$ be chosen such that $d(y_i, y_{i+1}) = \varepsilon$; $n\varepsilon$ being equal to d. Let $\varepsilon > 0$. Construct the sets C_i successively as follows:

$$C_1 = \overline{S(y_1, 2\varepsilon\sqrt{\eta})} ,$$

$$C_2 = \overline{\bigcup_{z_1 \in C_1} S(y_2, 2\sqrt{\eta}\, d(z_1, y_2))} ,$$

$$C_{j+1} = \overline{\bigcup_{z_j \in C_j} S(y_{j+1}, 2\sqrt{\eta}\, d(z_j, y_{j+1}))} ,$$

and let

$$\Delta_j = \sup_{z_j \in C_j} d(z_j, y_j) ,$$

$$\rho_{j-1} = \sup_{z_{j-1} \in C_{j-1}} d(z_{j-1}, y_j) ,$$

$$\mu_{j-1} = \inf_{z_{j-1} \in C_{j-1}} d(z_{j-1}, y_j) .$$

Then

$$\rho_{j-1} \leqq \varepsilon + \Delta_{j-1} ,$$
$$\mu_{j-1} \geqq \varepsilon - \Delta_{j-1} ,$$
$$\Delta_j \leqq 2\sqrt{\eta}\, \rho_{j-1} ,$$
$$\rho_0 = \varepsilon, \qquad \Delta_0 = 0 .$$

It follows from these equations that

$$\begin{aligned} \Delta_j &\leqq \varepsilon[2\sqrt{\eta} + (2\sqrt{\eta})^2 + \cdots (2\sqrt{\eta})^j] \\ &\leqq \varepsilon 2\sqrt{\eta}\, \frac{1}{1 - 2\sqrt{\eta}} \\ &\leqq 4\varepsilon\sqrt{\eta} \quad \text{for} \quad 2\sqrt{\eta} \leqq \tfrac{1}{2}, \qquad j = 1, 2, \cdots n . \end{aligned}$$

If ε_j is a typical distance from C_j to y_{j+1}, then

$$\varepsilon - \Delta_j \leqq \varepsilon_j \leqq \varepsilon + \Delta_j ,$$

so that

$$\varepsilon(1 - 4\sqrt{\eta}) \leqq \varepsilon_j \leqq \varepsilon(1 + 4\sqrt{\eta}) .$$

Now we can apply Theorem 3.8 and obtain a uniform estimate for x in C_{n-1}. Repeated application of Theorem 3.8 leads to

$$\phi(y, \lambda) \geqq \exp\{-d(1 + 4\sqrt{\eta})(1 + \delta(\varepsilon))\sqrt{2\lambda}\}$$
$$\times [1 - M_\rho \exp\{-\varepsilon(1 - 4\sqrt{\eta})(\eta - \rho - \delta(\varepsilon))\sqrt{2\lambda}\}]^n .$$

THEOREM 3.10. *Let z be a point in R_k, S the sphere $S = [x: d(x, z) < \delta]$. Let $\phi(x, \lambda)$ be the solution to the equation $L\phi = \lambda\phi$ in the exterior of S with $\phi = 1$ on the boundary of S. Then*

$$\liminf_{\lambda \to \infty} \frac{1}{\sqrt{2\lambda}} \log \phi(x, \lambda) \geqq -d(x, z)$$

uniformly over all x, z and δ, provided δ remains bounded away from zero and $d(x, z)$ remains bounded.

The proof follows at once from Theorem 3.9. One chooses η first and then ρ and ε suitably; n remains bounded for any $\varepsilon > 0$ if d is bounded.

Let G be a region with exterior E. B is the boundary of G. It is assumed that B is also the boundary of E.

LEMMA 3.11. *Let $x \in G$ and let $d = d(x, B)$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ and a point $z \in E$ such that*

(i) $d(x, z) \leqq d + \varepsilon$,

(ii) $[y: d(y, z) < \delta] \subset E$.

Moreover δ can be chosen uniformly when x varies over a bounded subset of G.

Proof: Let $x \in G \cup B$. Then there exists a y on B such that $d(x, B) = d(x, y)$. The sphere around y of radius $\frac{1}{2}\varepsilon$ has a non-empty intersection with E and there exists a sphere around some points z of some radius δ completely contained inside the intersection. Obviously $d(x, z) \leqq d(x, y) + \frac{1}{2}\varepsilon$. To show uniformity we point out that if x' is in $G \cup B$ with $d(x, x') < \frac{1}{2}\varepsilon$, then

$$\begin{aligned} d' = d(x', B) &\geqq d(x, B) - d(x, x') \\ &\geqq d - \tfrac{1}{2}\varepsilon \\ &\geqq d(x, z) - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\varepsilon \\ &= d(x, z) - \varepsilon , \end{aligned}$$

or

$$d(x, z) \leqq d' + \varepsilon .$$

In other words, the sphere of radius δ around z serves not only for x but for a neighborhood around x as well. A routine compactness argument using finite subcovers proves the lemma for compact subsets of $G \cup B$ or for bounded subsets of G.

Now we can prove the main theorem:

THEOREM 3.12. *Let G be any region with boundary B. B is assumed to be the boundary of the exterior E of G as well. If $\phi(x, \lambda)$ is the solution of the equation*

$$L\phi = \lambda\phi \quad \textit{for} \quad x \quad \textit{in} \quad G ,$$

$$\phi = 1 \qquad \textit{on} \quad B ,$$

then

$$\lim_{\lambda \to \infty} \left[-\frac{1}{\sqrt{2\lambda}} \log \phi(x, \lambda) \right] = d(x, B)$$

uniformly for x varying over bounded subsets of G.

The proof follows from Theorems 3.6, 3.10, and Lemma 3.11.

4. The Fundamental Solution

Let $F(t)$ be a nondecreasing function of t for $t \geqq 0$ with $0 \leqq F(t) \leqq 1$ and $F(0) = 0$. Let $\phi(\lambda)$ be the Laplace transform

$$\phi(\lambda) = \int_0^\infty e^{-\lambda\tau}\, dF(\tau) . \tag{4.1}$$

We assume that the $\{\phi_\alpha(\lambda)\}$ are Laplace transforms of $\{F_\alpha(t)\}$ and that

$$\lim_{\lambda \to \infty} \left[-\frac{1}{\sqrt{2\lambda}} \log \phi_\alpha(\lambda) \right] = d_\alpha , \tag{4.2}$$

where the d_α are such that $0 \leqq d_\alpha \leqq D < \infty$. We want to conclude that if the limit in (4.2) is uniform with respect to α, then

$$\lim_{t \to 0} \left[-2t \log F_\alpha(t) \right] = d_\alpha^2 \tag{4.3}$$

uniformly in α.

Initially we assume that (4.2) holds uniformly and that in addition $0 < d \leqq d_\alpha \leqq D < \infty$. The following lemmas are conclusions regarding $F_\alpha(t)$.

LEMMA 4.1. *For every $\varepsilon > 0$ there exists a constant $A_\varepsilon < \infty$ such that, for all $t \geqq 0$,*

$$F_\alpha(t) \leqq A_\varepsilon \exp \left\{ -\frac{(1 - \varepsilon)}{2t} d_\alpha^2 \right\} .$$

Proof: Since $0 < d \leqq d_\alpha \leqq D < \infty$ we can conclude, from the fact that (4.2) holds uniformly, that for every $\varepsilon > 0$ there exists a constant A_ε such that for all $\lambda \geqq 0$

$$\phi_\alpha(\lambda) \leqq A_\varepsilon \exp \{ -\sqrt{1 - \varepsilon}\, \sqrt{2\lambda}\, d\alpha \} . \tag{4.4}$$

It follows from (4.1) that

$$\phi_\alpha(\lambda) \geqq e^{-\lambda t} F_\alpha(t) ,$$

or

$$F_\alpha(t) \leqq e^{\lambda t} \phi_\alpha(\lambda) . \tag{4.5}$$

Choosing $\lambda = (1 - \varepsilon) d_\alpha^2 / 2t$ and combining (4.5) and (4.4) we get

$$F_\alpha(t) \leqq A_\varepsilon \exp \left\{ -\frac{(1 - \varepsilon)}{2t} d_\alpha^2 \right\} .$$

COROLLARY 4.2.

$$\limsup_{t \to 0} 2t \log F_\alpha(t) \leqq -d_\alpha^2$$

uniformly in α.

LEMMA 4.3. *For every* $k > 1$,

$$\int_{kt}^{\infty} \exp\left\{-\frac{d_\alpha^2 \tau}{2t^2}\right\} F_\alpha(\tau)\, d\tau \leqq \frac{2A_\varepsilon t^2}{\varepsilon d^2} \exp\left\{-\frac{(1-\varepsilon)d_\alpha^2}{2t} - \frac{\rho_k}{t}\right\},$$

where $\rho_k = \frac{1}{2} d^2(k + 1/k - 2)$.

Proof: Using Lemma 4.1, we have

$$\int_{kt}^{\infty} \exp\left\{-\frac{d_\alpha^2 \tau}{2t}\right\} F_\alpha(\tau)\, d\tau \leqq A_\varepsilon \int_{kt}^{\infty} \exp\left\{-\frac{d_\alpha^2 \tau}{2t^2}\right\} \exp\left\{-\frac{(1-\varepsilon)d_\alpha^2}{2t}\right\} d\tau$$

$$= A_\varepsilon t \int_k^{\infty} \exp\left\{-\frac{1}{2t} d_\alpha^2 \left[u + \frac{(1-\varepsilon)}{u}\right]\right\} du$$

$$= A_\varepsilon t \int_k^{\infty} \exp\left\{-\frac{\varepsilon d_\alpha^2}{2t}\right\} \exp\left\{-\frac{(1-\varepsilon)d_\alpha^2}{2t}\left[u + \frac{1}{u}\right]\right\} du$$

$$\leqq \frac{2A_\varepsilon t^2}{\varepsilon d^2} \exp\left\{-\frac{(1-\varepsilon)d_\alpha^2}{t} - \frac{\rho_k}{t}\right\}.$$

LEMMA 4.4. *For every* $k' < 1$,

$$\int_0^{k't} \exp\left\{-\frac{d_\alpha^2 \tau}{2t^2}\right\} F_\alpha(\tau)\, d\tau \leqq \frac{2A_\varepsilon t^2}{\varepsilon d^2} \exp\left\{-\frac{(1-\varepsilon)d_\alpha^2}{t} - \frac{\rho_{k'}}{t}\right\},$$

where $\rho_{k'} = \frac{1}{2} d^2[k' + 1/k' - 2]$.

The proof is similar to that of Lemma 4.3.

LEMMA 4.5.

$$\lim_{t \to 0} \left[-2t \log F_\alpha(t)\right] = d_\alpha^2$$

uniformly in α.

Proof: Because of (4.2), there exists, for every $\varepsilon > 0$, a constant B_ε such that for all $\lambda \geqq 0$

$$\phi_\alpha(\lambda) \geqq B_\varepsilon \exp\{-(1+\varepsilon)\sqrt{2\lambda}\, d_\alpha\}. \tag{4.6}$$

However,

$$\phi_\alpha(\lambda) = \lambda \int_0^{\infty} e^{-\lambda\tau} F_\alpha(\tau)\, d\tau$$

$$= \lambda\left[\int_0^{k't} + \int_{k't}^{kt} + \int_{kt}^{\infty}\right] e^{-\lambda\tau} F_\alpha(\tau)\, d\tau. \tag{4.7}$$

Therefore, substituting $\lambda = d_\alpha^2/2t^2$ and combining (4.6) and (4.7) with Lemmas 4.3 and 4.4, we have

$$\int_{k't}^{kt} \exp\left\{-\frac{d_\alpha^2 \tau}{2t^2}\right\} F_\alpha(\tau)\, d\tau \geqq C_\varepsilon t^2 \exp\left\{\frac{(1+\varepsilon)\, d_\alpha^2}{t}\right\}\left(1 - k_\varepsilon \exp\left\{\frac{2\varepsilon D^2 - \rho}{t}\right\}\right), \tag{4.8}$$

where $\rho = \min(\rho_k, \rho_{k'})$.

For a given k and k' with $k' < 1 < k$ one can find ε small enough so that $2\varepsilon D^2 - \rho < 0$. Therefore, we conclude from (4.8) that, for every k, k' with $k' < 1 < k$,

$$\liminf_{t \to 0} t \log\left[\int_{k't}^{kt} \exp\left\{-\frac{d_\alpha^2}{2t^2}\right\} F_\alpha(\tau)\, d\tau\right] \geqq -d_\alpha^2 \tag{4.9}$$

uniformly in α.

On the other hand,

$$\int_{k't}^{kt} \exp\left\{-\frac{d_\alpha^2 \tau}{2t^2}\right\} F_\alpha(\tau)\, d\tau \leqq F_\alpha(kt) \exp\left\{-\frac{k' d_\alpha^2}{2t}\right\}$$

which with (4.9) yields

$$\liminf_{t \to 0} t \log F_\alpha(kt) \geqq -d_\alpha^2 + \tfrac{1}{2} k' d_\alpha^2,$$

or

$$\liminf_{t \to 0} 2t \log F_\alpha(t) \geqq -2k d_\alpha^2 + kk' d_\alpha^2$$

uniformly in α. Since k and k' can be chosen close to 1, we have

$$\liminf_{t \to 0} 2t \log F_\alpha(t) \geqq -d_\alpha^2$$

uniformly in α. This is one part of the lemma and the other part is Corollary 4.2.

Now we drop the assumption that $d > 0$. We assume only that $0 \leqq d_\alpha \leqq D$.

LEMMA 4.6. *Lemma* 4.5 *still holds if we assume only that* $0 \leqq d_\alpha \leqq D$.

Proof: It is of course enough to show that if, for $\lambda \geqq \lambda_0$,

$$-\varepsilon \leqq -\frac{1}{\sqrt{2\lambda}} \log \phi_\alpha(\lambda) \leqq \varepsilon ,$$

then there exists a t_0 and a function $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$-\eta \leqq -2t \log F_\alpha(t) \leqq \eta .$$

Since $0 \leqq \phi_\alpha(\lambda) \leqq 1$ and $0 \leqq F_\alpha(t) \leqq 1$, one side is trivial. If

$$\phi_\alpha(\lambda) \geqq e^{-\varepsilon\sqrt{2\lambda}} \qquad \text{for} \qquad \lambda \geqq \lambda_0 ,$$

we shall show that for ε small enough there exist η and t_0 such that

$$F_\alpha(t) \geqq e^{-\eta/t} \qquad \text{for} \qquad t \leqq t_0 .$$

To prove this we notice that

$$\phi_\alpha(\lambda) \leqq F_\alpha(t) + e^{-\lambda t} ,$$

or

$$F_\alpha(t) \geqq \phi_\alpha(\lambda) - e^{-\lambda t}$$
$$\geqq e^{-\varepsilon\sqrt{2\lambda}} - e^{-\lambda t} \qquad \text{for} \qquad \lambda \geqq \lambda_0 .$$

Choose $\lambda = 2/t^2$. If $t \leqq t_0 = \sqrt{2/\lambda_0}$ and $\varepsilon < \frac{1}{2}$, then

$$f_\alpha(t) \geqq e^{-2\varepsilon/t}(1 - e^{-1/t})$$

which concludes the proof.

We have, therefore, proved the following theorem:

THEOREM 4.7. *Let* $\phi_\alpha(\lambda) = \int_0^\infty e^{-\lambda t}\, dF_\alpha(\tau)$. *If*

$$\lim_{\lambda\to\infty} \left[-\frac{1}{\sqrt{2\lambda}} \log \phi_\alpha(\lambda) \right] = d_\alpha$$

uniformly in α *and if* $0 \leqq d_\alpha \leqq D < \infty$, *then*

$$\lim_{t\to 0} \left[-2t \log F_\alpha(t) \right] = d_\alpha^2$$

uniformly in α.

Let $G \subset R_k$ be a region with a boundary B. It is assumed that B is the boundary of the exterior of G as well. Let S be the cylinder in $R_k \times (0, \infty)$ with base G. That is,

$$S = G \times (0, \infty) .$$

The boundary of S is $\{B \times (0, \infty)\} \cup \{G \times \{0\}\}$. We consider the solution $u(x, t)$ to the equation

$$\text{(4.10)} \qquad \begin{aligned} u_t - Lu &= 0 && \text{for} \quad (x, t) \in G \times (0, \infty) , \\ u(x, 0) &= 0 && \text{for} \quad x \in G, \\ u(x, t) &= 1 && \text{for} \quad x \in B, \quad t > 0 . \end{aligned}$$

The solution to the equation exists and has the properties:

$u(x, t)$ is nondecreasing in t for $t \geqq 0$,
$0 \leqq u(x, t) \leqq 1$.

THEOREM 4.8.

$$\limsup_{t\to 0} 2t \log u(x, t) \leqq -d^2(x, B)$$

uniformly over all regions G and points x in G so long as $d(x, B)$ *remains bounded.*

Proof: If $\phi(x, \lambda) = \int_0^\infty e^{-\lambda\tau} u(x, d\tau)$, then $\phi(x, \lambda)$ satisfies the equation

$$\text{(4.11)} \qquad \begin{aligned} L\phi &= \lambda\phi && \text{for} \quad x \in G , \\ \phi &= 1 && \text{for} \quad x \in B . \end{aligned}$$

From Theorem 3.6, there exists for every $\rho > 0$, a $C_\rho < \infty$ such that

$$\phi(x, \lambda) \leqq C_\rho \exp\{-(1 - \rho)\sqrt{2\lambda}\, d(x, B)\}$$

so long as $d(x, B)$ remains bounded. From equation (4.5), we have

$$u(x, t) \leqq e^{\lambda t}\phi(x, \lambda)$$

for every $\lambda \geqq 0$. Choosing $\lambda = (1 - \rho)^2 d^2(x, B)/2t^2$,

$$u(x, t) \leqq C_\rho \exp\left\{\frac{(1 - \rho)^2 d^2(x, B)}{2t}\right\}$$

which proves the theorem.

THEOREM 4.9. *The fundamental solution* $p(t, x, y)$ *of Theorem* 2.1 *has the property*

$$\limsup_{t\to 0} 2t \log p(t, x, y) \leqq -d^2(x, y)$$

uniformally over all x, y *such that* $d(x, y)$ *is bounded.*

Proof: Let $d(x,y) = d$. Define, for any $\varepsilon > 0$,

$$G = [z: d(x, z) < d - \varepsilon],$$

$$B = [z: d(x, z) = d - \varepsilon].$$

Then $p(t, x, y)$ satisfies the equation

$$\begin{aligned} p_t - Lp &= 0 \quad \text{for} \quad (x, t) \text{ in } G \times (0, \infty), \\ p(0, x, y) &= 0 \quad \text{for} \quad x \in G, \\ p(t, x, y) &= p(t, b, y) \quad \text{for} \quad b \in B. \end{aligned} \tag{4.12}$$

If $u(x, t)$ is the solution to

$$\begin{aligned} u_t - Lu &= 0 \quad \text{for} \quad (x, t) \text{ in } G \times (0, \infty), \\ u &= 0 \quad \text{for} \quad x \in G \text{ and } t = 0, \\ u &= 1 \quad \text{for} \quad x \in B \text{ and } t > 0, \end{aligned} \tag{4.13}$$

then clearly

$$p(t, x, y) \leqq \left[\sup_{\substack{0 \leqq \tau < \infty \\ b \in B}} p(\tau, b, y)\right] u(x, t). \tag{4.14}$$

Since $d(b, y) \geqq \varepsilon$, we have from property (iv) of Theorem 2.1

$$\sup_{0 < \tau < \infty} \sup_{b \in B} p(\tau, b, y) \leqq M_\varepsilon < \infty. \tag{4.15}$$

Hence, combining (4.14) and (4.15) with Theorem 4.8, we obtain

$$\lim_{t \to 0} 2t \log p(t, x, y) \leqq -(d - \varepsilon)^2.$$

Since ε is arbitrary, this proves the theorem. However, apparently the uniformity holds only in regions of the form $0 < \varepsilon \leqq d(x,y) \leqq D < \infty$. But the direct estimate in Theorem 2.1,

$$p(t, x, y) \leqq C t^{-k/2} \exp\left\{\frac{-\alpha \|x - y\|^2}{2t}\right\},$$

establishes the uniformity over all x, y such that $d(x, y)$ remains bounded.

We now proceed to obtain a lower bound on $p(t, x, y)$ for small t.

THEOREM 4.10. *There exists a function $\psi(\varepsilon)$ such that $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$ and*

$$\liminf_{t \to 0} \inf_{x,y: d(x,y) \leqq \varepsilon} 2t \log p(t, x, y) \geqq -\psi(\varepsilon).$$

Proof: Let $s > 0$. Define G_s and its boundary B_s by

$$G_s = [z: d(y, z) > s],$$

$$B_s = [z: d(y, z) = s].$$

Let $u_s(x, t)$ be the solution to the equation

$$\begin{aligned} u_t - Lu &= 0 \quad \text{in} \quad G_s \times (0, \infty), \\ u &= 1 \quad \text{on} \quad B_s \times (0, \infty), \\ u &= 0 \quad \text{on} \quad G_s \times \{0\}. \end{aligned} \tag{4.16}$$

Let $\phi_s(x, \lambda) = \int_0^\infty e^{\lambda\tau} u_s(x, d\tau)$. Then $\phi_s(x, \lambda)$ satisfies the equation

$$\begin{aligned} L\phi &= \lambda\phi \quad \text{in} \quad G_s, \\ \phi &= 1 \quad \text{on} \quad B_s. \end{aligned} \tag{4.17}$$

From Theorem 3.9,

$$\begin{aligned} \phi_s(x, \lambda) \geqq{}& \exp\{-d(1 + 4\sqrt{\eta})(1 + \delta(\varepsilon))\sqrt{2\lambda}\} \\ &\times [1 - M_\rho \exp\{-\varepsilon(1 - 4\sqrt{\eta})(\eta - \rho - \delta(\varepsilon))\sqrt{2\lambda}\}]^n, \end{aligned} \tag{4.18}$$

where $d = d(x,y)$, ε, η, ρ are arbitrary, $n\varepsilon = d$, with the restriction that $s \geqq 4\varepsilon\sqrt{\eta}$. Since

$$u_s(x, t) \geqq \phi_s(x, \lambda) - e^{-\lambda t}$$

for every λ, let us choose $\lambda = 8/t^2$, $\varepsilon = s$ and η small enough. We further put $s = \sqrt{t}$. Then using (4.18) we get

$$u_{\sqrt{t}}(x, t) \geqq \exp\{-\frac{4d}{t}(1 + \theta)\}(1 - M_\rho e^{-C/\sqrt{t}})^{d/\sqrt{t}} - e^{-8/t}, \tag{4.19}$$

where θ is a small number which can be chosen arbitrarily small. This function is defined for x such that $d(x, y) \geqq s = \sqrt{t}$. It is clear from (4.19) that if $\varepsilon \leqq 1$

$$\liminf_{t \to 0} \inf_{\substack{x,y: d(x,y) \geqq \sqrt{t} \\ d(x,y) \leqq \varepsilon}} t \log u_{\sqrt{t}}(x, t) \geqq -4\varepsilon. \tag{4.20}$$

We now consider the solution $u_s(x, t_0, t)$ to the equation

$$\begin{aligned} u_t - Lu &= 0 \quad \text{in} \quad G_s \times (0, \infty), \\ u &= 0 \quad \text{on} \quad B_s \times (0, t_0), \\ u &= 1 \quad \text{on} \quad B_s \times (t_0, \infty), \\ u &= 0 \quad \text{on} \quad G_s \times \{0\}. \end{aligned} \tag{4.21}$$

It is clear that

$$u_s(x, t_0, t) = 0 \quad \text{for} \quad t \leqq t_0,$$
$$u_s(x, t_0, t) = u_s(x, t - t_0) \quad \text{for} \quad t > t_0.$$

We are interested in $u_{\sqrt{t}}(x, \frac{1}{2}t, t) = u_{\sqrt{t}}(x, \frac{1}{2}t)$. From (4.20) we deduce

$$\liminf_{t \to 0} \inf_{\substack{(x,y): d(x,y) \geqq \sqrt{t} \\ d(x,y) \leqq \varepsilon}} t \log u_{\sqrt{t}}(x, \tfrac{1}{2}t, t) \geqq -8\varepsilon. \tag{4.22}$$

Going back to $p(t, x, y)$ we see that it satisfies an equation similar to (4.21) in $G_s \times (0, \infty)$ with different boundary values. However, since the solution $u_s(x, t_0, t)$ depends only on the boundary values on $B_s \times (0, \infty)$ for $0 < \tau < t$ we conclude by direct comparison that

$$p(t, x, y) \geqq u_s(x, t_0, t) \inf_{\substack{b \in B_s \\ t_0 \leqq \tau \leqq t}} p(\tau, b, y).$$

Since t_0 and t are arbitrary, we choose $t_0 = \frac{1}{2}t$ and $s = \sqrt{t}$. Therefore,

$$p(t, x, y) \geqq u_{\sqrt{t}}(x, \tfrac{1}{2}t, t) \inf_{\substack{b \in B_{\sqrt{t}} \\ \frac{1}{2}t \leqq \tau \leqq t}} p(\tau, b, y). \tag{4.23}$$

For t sufficiently small we have from property (v) of Theorem 2.1

$$\inf_{\substack{b \in B_{\sqrt{t}} \\ \frac{1}{2}t \leqq \tau \leqq t}} p(\tau, b, y) \geqq \gamma > 0.$$

This with (4.22) and (4.23) yields

$$\liminf_{t \to 0} \inf_{\substack{x,y: d(x,y) \leqq \varepsilon \\ d(x,y) \geqq \sqrt{t}}} 2t \log p(t, x, y) \geqq -8\varepsilon.$$

On the other hand, if $d(x, y) \leqq \sqrt{t}$, again from property (v) of Theorem 2.1,

$$\liminf_{t \to 0} \inf_{x,y: d(x,y) \leqq \sqrt{t}} 2t \log p(t, x, y) \geqq 0.$$

This completes the proof.

THEOREM 4.11. *Let $\varepsilon > 0$ and let G_y be the set $[z: d(y, z) < \varepsilon]$. If $k(x, t)$ is defined as*

$$k(x, t, y) = \int_{G_y} p(t, x, z)\, dz,$$

then

$$\liminf_{t \to 0} 2t \log k(x, t, y) \geqq -d^2(x, y)$$

uniformly over all x, y such that $d(x, y)$ is bounded.

Proof: Let

$$H_y = [z: d(y, z) > \tfrac{1}{2}\varepsilon],$$
$$B_y = [z: d(y, z) = \tfrac{1}{2}\varepsilon].$$

From property (iv) of Theorem 2.1 it follows that

$$k(x, t, y) \to 1 \quad \text{as} \quad t \to 0 \tag{4.24}$$

uniformly for x on B_y.

Assume that $u(x, t)$ solves the equation

$$\begin{aligned} u_t - Lu &= 0 \quad \text{in} \quad H_y \times (0, \infty), \\ u &= 1 \quad \text{on} \quad B_y \times (0, \infty), \\ u &= 0 \quad \text{on} \quad H_y \times \{0\}. \end{aligned} \tag{4.25}$$

Then the Laplace transform $\phi(x, \lambda)$ solves the equation

$$L\phi = \lambda\phi \quad \text{for} \quad x \quad \text{in} \quad H_y,$$
$$\phi = 1 \quad \text{on} \quad B_y.$$

From Theorem 3.12 it follows that

$$\lim_{t \to \infty} \left[-\frac{1}{\sqrt{2\lambda}} \log \phi(x, \lambda) \right] = d(x, B_y).$$

Moreover, from Theorems 3.6 and 3.10 it is clear that the above limit is uniform whenever $d(x, y)$ remains bounded. Of course $\phi(x, \lambda)$ is defined only when $d(x, y) \geqq \frac{1}{2}\varepsilon$. Now from Theorem 4.7 we obtain

$$\lim_{t \to \infty} 2t \log u(x, t) \geqq -d^2(x, B_y) \geqq -d^2(x, y) \tag{4.26}$$

uniformly whenever $d(x, y)$ remains bounded. Standard comparison of $k(x, t, y)$ which satisfies an equation similar to (4.25) with $u(x, t)$ shows that

$$k(x, t, y) \geqq u(x, t) \inf_{\substack{0 \leqq \tau \leqq t \\ b \in B_y}} k(b, \tau, y). \tag{4.27}$$

Inequalities (4.26) and (4.27) complete the proof insofar as the region $d(x, y) \geqq \frac{1}{2}\varepsilon$ is concerned. However, property (iv) of Theorem 2.1 shows that

$$k(x, t, y) \to 1 \quad \text{as} \quad t \to 0$$

uniformly over the set $d(x, y) \leqq \frac{1}{2}\varepsilon$.

THEOREM 4.12.

$$\liminf_{t \to 0} 2t \log p(t, x, y) \geqq -d^2(x, y)$$

uniformly over all x, y such that $d(x, y)$ is bounded.

Proof: If G_y is the set $[z : d(y, z) \leqq \varepsilon]$, then

$$p(t, x, y) = \int p([1 - \delta]t, x, z) p(\delta t, z, y) \, dz$$

$$\geqq \int_{G_y} p([1 - \delta]t, x, z) p(\delta t, z, y) \, dz$$

$$\geqq k(x, [1 - \delta]t, y) \inf_{z \in G_y} p(\delta t, z, y) .$$

Therefore, for every $\varepsilon > 0$ and $\delta > 0$, applying Theorems 4.10 and 4.11 we get

$$\liminf_{t \to 0} 2t \log p(t, x, y) \geqq - \left[\frac{d^2(x, y)}{1 - \delta} + \frac{\psi(\varepsilon)}{\delta} \right] .$$

Since δ is arbitrary, let us choose

$$\delta = \frac{[\psi(\varepsilon)]^{1/2}}{d(x, y) + [\psi(\varepsilon)]^{1/2}} .$$

Then

$$\liminf_{t \to 0} 2t \log p(t, x, y) \geqq -[d(x, y) + [\psi(\varepsilon)]^{1/2}] .$$

Since ε is arbitrary and $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$,

$$\liminf_{t \to 0} 2t \log p(t, x, y) \geqq -d^2(x, y) .$$

Since $d(x, y)$ is bounded, the only possible trouble regarding uniformity can occur when $\delta \to 1$ or $d(x, y)$ is very small. But then Theorem 4.10 can be used directly to show that it is uniform even around $d(x, y) = 0$.

We have finally proved

THEOREM 4.13.

$$\lim_{t \to 0} \left[-2t \log p(t, x, y) \right] = d^2(x, y)$$

uniformly in x, y such that $d(x, y)$ is bounded.

Remarks. The assumption in Theorem 3.12 that B is the common boundary of G and of its exterior can be weakened further. It suffices to assume that, for a dense set $B_0 \subset B$, every sufficiently small sphere S around $b_0 \in B_0$ is disconnected by B.

We can consider equation (2.3) in a domain $G \subset R_k$ with the extra condition that the solution is zero on the boundary. Then there is a formula similar to Theorem 2.2, where $d(x, y)$ is replaced by $d_G(x, y)$ which is the length of the shortest path from x to y not leaving G.

These results are best proved using probabilistic methods and are therefore left to [3].

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