This is under the assumption of (5.1); Theorem 2 follows immediately. Setting $\Omega = \Omega_1 N^{\alpha-1}$ gives Corollary 2.1 and (1.5). For Theorem 3, we remark that (1.6) implies

$$\sum_1^N E(|x_k|^p) = p\int_0^\infty \lambda^{p-1} \sum_1^N P(|x_k| > \lambda)\, d\lambda$$

$$\leqq pCN^\alpha \int_0^\infty \lambda^{p-1} e^{-\beta\lambda^\varepsilon}\, d\lambda$$

$$= \frac{pCN^\alpha}{\varepsilon}\left(\frac{1}{\beta}\right)^{p/\varepsilon}\Gamma\left(\frac{p}{\varepsilon}\right) = \Omega_p N .$$

We apply (5.4) with $p = \log N$. Thus if $\Omega_p \geqq 1$,

$$\Omega_p^{1/(p+1)} \leqq \Omega_p^{1/p} = \left(\frac{pCN^\alpha}{\varepsilon}\right)^{1/p}\left(\frac{1}{\beta}\right)^{1/\varepsilon}\Gamma\left(\frac{p}{\varepsilon}\right)^{1/p} = O((\log N)^{1/\varepsilon}),$$

$$\frac{1}{N^{(p-2)/2(p+1)}} = \frac{1}{\sqrt{N}}\, N^{3/2(p+1)} \leqq \frac{N^{3/2p}}{\sqrt{N}} = O\left(\frac{1}{\sqrt{N}}\right),$$

$$\left(\frac{\Omega_p}{N^{(p-5)/4}}\right)^{2/(p+1)} \sqrt{\log N} \leqq \frac{\Omega_p^{2/p} N^{3/p}\sqrt{\log N}}{\sqrt{N}} = O\left(\frac{(\log N)^{2/\varepsilon+1/2}}{\sqrt{N}}\right) \to 0$$

and Theorem 3 follows.

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Diffusion Processes in a Small Time Interval*

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1. Introduction

Let $X(t)$ be a diffusion process in k dimensions with diffusion coefficients $\{a_{ij}(x)\}$. The transition probability densities of the homogeneous Markov process $X(t)$ are given by the fundamental solution $p(t, x, y)$ of the equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\sum a_{ij}(x)\,\frac{\partial^2 p}{\partial x_i\,\partial x_j}\,. \tag{1.1}$$

Our interest is in the behavior of the process $X(t)$ for small t. For that purpose we introduce a small parameter $\varepsilon > 0$ and define a process $X_\varepsilon(t) = X(\varepsilon t)$. We consider in detail the process $X_\varepsilon(t)$ in a fixed time interval $[0, T]$. This of course corresponds to studying the behavior, as $\varepsilon \to 0$, of the diffusion processes corresponding to the equations

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\varepsilon\sum a_{ij}(x)\,\frac{\partial^2 p}{\partial x_i\,\partial x_j}\,.$$

A similar analysis for Brownian motion was carried out by M. Schilder [4] and such an analysis has been used by Donsker in [2]. The same approach in connection with processes with independent increments was considered by the author, [6].

Section 2 outlines the contents of the paper and describes some preliminary material. The later sections contain the actual proofs.

2. Preliminaries and Summary

We denote by L_ε the operator

$$L_\varepsilon f = \tfrac{1}{2}\varepsilon\sum a_{ij}(x)\,\frac{\partial^2 f}{\partial x_i\,\partial x_j} \tag{2.1}$$

acting on smooth functions on R_k. The following assumptions are made regarding the coefficients $\{a_{ij}(x)\}$:

(A) $\{a_{ij}(x)\}$ is a symmetric positive definite matrix for each x.

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(B) The coefficients satisfy a uniform Hölder condition:

$$|a_{ij}(x) - a_{ij}(y)| \leqq M |x - y|^{\alpha} .$$

(C) The matrices $\{a_{ij}(x)\}$ satisfy a uniform ellipticity condition:

$$\alpha_1 \sum \xi_j^2 \leqq \sum a_{ij}(x)\xi_i\xi_j \leqq \alpha_2 \sum \xi_j^2$$

for some $0 < \alpha_1 \leqq \alpha_2 < \infty$.

Under these assumptions there exists a fundamental solution $p_\varepsilon(t, x, y)$ to the equation

(2.2) $$\frac{\partial p}{\partial t} = \tfrac{1}{2}\varepsilon \sum a_{ij}(x) \frac{\partial^2 p}{\partial x_i \, \partial x_j} .$$

Moreover, if we denote L_ε for $\varepsilon = 1$ by L, and the corresponding $p_\varepsilon(t, x, y)$ by $p(t, x, y)$, it is obvious that

(2.3) $$p_\varepsilon(t, x, y) = p(\varepsilon t, x, y) .$$

Let us choose and fix a $T < \infty$. We denote by Ω the space of all continuous functions $\omega(t)$ on $[0, T]$ with values in R_k. The topology in Ω is that of uniform convergence. The measurable sets are the Borel sets. We denote by Ω_x all functions $\omega \in \Omega$ with $\omega(0) = x$, where x is a point in R_k. For any set $A \subset \Omega$, A_x will be the intersection of A and Ω_x, $\|\omega\|$ will denote the sup norm.

For every $\varepsilon > 0$, $p_\varepsilon(t, x, y)$ satisfies the Chapman-Kolmogorov equations (semigroup property)

(2.4) $$p_\varepsilon(t + s, x, y) = \int p_\varepsilon(t, x, z) p_\varepsilon(s, z, y) \, dz .$$

We can therefore use these functions as probability densities and construct a Markov process. For each $\varepsilon > 0$ they lead to a measure P_x^ε on Ω corresponding to the process starting at time $t = 0$ at the point x. In other words,

(2.5) $$P_x^\varepsilon[\Omega_x] = 1 .$$

If $A \subset \Omega$ is a cylinder set of the form

$$A = \{\omega : \omega(t_1), \cdots \omega(t_n) \in B\} ,$$

where $0 < t_1 < t_2 \cdots < t_n \leqq T$ and B is a measurable subset of the n-fold product of R_k, then $P_x^\varepsilon[A]$ is defined as

(2.6) $$P_x^\varepsilon[A] = \int_B p_\varepsilon(t_1, x, y_1) p_\varepsilon(t_2 - t_1, y_1, y_2) p_\varepsilon(t_n - t_{n-1}, y_{n-1}, y_n) \, dy_1 \cdots dy_n .$$

The relation (2.6) completely determines P_x^ε on Ω. The family $\{P_x^\varepsilon\}$ is referred to as the Markov process corresponding to the operator L_ε. For more details and for further properties of $\{P_x^\varepsilon\}$, such as the strong Markov property, the reader is referred to Dynkin [3].

We are interested in the behavior of P_x^ε as $\varepsilon \to 0$. For this purpose we introduce a functional $I(\omega)$ on Ω. Let us denote by $a^{-1}(x)$ the matrix inverse of the coefficient matrix $\{a_{ij}(x)\}$ at the point x. If θ is a vector, then $\theta a^{-1}(x)\theta$ will denote the quadratic form using the matrix $a^{-1}(x)$. For any $\omega \in \Omega$ which is differentiable, $\dot\omega(\tau)$ will be the vector $d\omega(\tau)/d\tau$. We now define $I(\omega)$ as follows:

If ω is absolutely continuous with a square integrable derivative on $[0, T]$, then

(2.7a) $$I(\omega) = \tfrac{1}{2}\int_0^T [\dot\omega(\tau) a^{-1}(\omega(\tau))\dot\omega(\tau)] \, d\tau ,$$

otherwise,

(2.7b) $$I(\omega) = +\infty .$$

We have the following theorem concerning $I(\omega)$.

THEOREM 2.1. *The function $I(\omega)$ is lower semicontinuous on all of Ω. Moreover, for any finite constant N, the set $\{\omega : I(\omega) \leqq N\}$ is an equicontinuous family of continuous functions on $[O, T]$.*

The proof is straightforward and is omitted.

We next introduce the global Riemannian distance $d(x, y)$ on R_k. For smooth functions $\omega(\tau)$ we first define the length

(2.8) $$l(\omega) = \int_0^T \sqrt{\dot\omega(\tau) a^{-1}(\omega(\tau))\dot\omega(\tau)} \, d\tau ,$$

and then we define

$$d(x, y) = \inf_{\omega : \substack{\omega(0)=x \\ \omega(T)=y}} l(\omega) .$$

It is clear that d is independent of T and is the natural geodesic distance; $l(\omega)$ is of course the geodesic length.

There is a relation between $I(\omega)$ and $l(\omega)$. The geodesic length $l(\omega)$ is independent of the parametrization of the curve, whereas $I(\omega)$ depends on the parametrization. For, a class of ω all representing the same curve $I(\omega)$ is minimum when the curve is parametrized so as to cover equal lengths in equal time. This corresponds to the fact that for the minimizing parametrization the integrand in (2.8) is a constant. We therefore obtain

LEMMA 2.2. *For* $0 \leqq \alpha < \beta \leqq T$,

$$\inf_{\omega:\,\substack{\omega(\alpha)=x\\ \omega(\beta)=y}} I(\omega) = \frac{d^2(x, y)}{2(\beta-\alpha)} .$$

An obvious extension is

LEMMA 2.3. *For* $0 \leqq t_1 < t_2 < t_3 \cdots < t_n \leqq T$,

$$\inf_{\omega:\,\substack{\omega(t_j)=x_j\\ j=1,2,\cdots n}} I(\omega) = \frac{1}{2}\sum_{j=1}^{n-1} \frac{d^2(x_{j+1}, x_j)}{t_{j+1}-t_j} .$$

We are now ready to state the main theorem of Section 3.

THEOREM 2.4. *Let* $C \subset \Omega$ *be closed and* $G \subset \Omega$ *be open. Then*

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C] \leqq -\inf_{\omega\in C_x} I(\omega) ,$$

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[G] \geqq -\inf_{\omega\in G_x} I(\omega) .$$

In proving Theorem 2.4 the following theorem proved in [7] plays an important role.

THEOREM 2.5.

$$\lim_{t\to 0} -2t \log p(t, x, y) = d^2(x, y)$$

uniformly over all x, y *such that* $d(x, y)$ *is bounded.*

The rest of this paper, namely Sections 4, 5, 6 are applications of Theorem 2.4. In Section 4 the following problem is considered: Let $D \subset R_k$ be an open subset with a boundary B. Let us consider the solution $p_D(t, x, y)$ of equation (1.1) but with the boundary condition

$$p_D(t, x, y) \to 0 \qquad \text{as} \qquad x \to B .$$

In other words, the process is killed on reaching the boundary B of D. We want to consider the behavior of

(2.9) $$\lim_{t\to 0} \frac{p_D(t, x, y)}{p(t, x, y)}$$

for points x, y in D. We introduce the distance

$$d_D(x, y) = \inf_{\omega:\,\substack{\omega(0)=x\\ \omega(T)=y\\ \omega(\tau)\in D \text{ for } 0\leqq\tau\leqq T}} l(\omega) .$$

For suitably smooth regions D it is shown that

$$\lim_{t\to 0} -2t \log p_D(t, x, y) = d_D^2(x, y) .$$

Moreover, the ratio (2.9) behaves as $t \to 0$ as follows: If the geodesics from x to y do not leave D, the ratio goes to 1. Otherwise it goes to 0. This is a rough form of the result. The precise version will be found in Section 4.[1] Section 5 is concerned with some applications to certain differential equations. Section 6 deals with the case when the operator L has an additional first order term. Essentially the same results can be obtained in this case. In fact, they are deduced from the present case.

3. Behavior of $\{P_x^\varepsilon\}$

Let $0 = t_0 < t_1 < t_2 \cdots < t_n = T$ be some time points. We shall denote by π this partition of the interval $[O, T]$. T_π will denote the map from Ω to the $(n + 1)$-fold product of R_k defined by

(3.1) $$T_\pi\omega = \{\omega(t_0), \omega(t_1), \cdots, \omega(t_n)\} .$$

For any set A in the range of T_π the inverse image $T_\pi^{-1}A$ is defined in the usual manner. We then have from (2.6)

(3.2) $$P_y^\varepsilon[T_\pi^{-1}A] = \int_{A_y} \prod_{j=1}^{n} p_\varepsilon(t_j - t_{j-1}, y_{j-1}, y_j)\, dy_j ,$$

where A_y is the section of A with $y_0 = y$.

LEMMA 3.1. *Let* $C \subset \Omega$ *be a closed set of the form* $C = T_\pi^{-1}A$, *where* A *is closed. Then*

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C] \leqq -\inf_{\omega\in C_x} I(\omega) .$$

Proof: First let us suppose that A is closed and bounded. Then

(3.3) $$P_y^\varepsilon[C] = \int_{A_y} \prod_{j=1}^{n} p_\varepsilon(t_j - t_{j-1}, y_{j-1}, y_j)\, dy_j .$$

[1] A more detailed investigation of this behavior for the case of Brownian motion was carried out by Ciesielski [1].

From Theorem 2.5 and Lemma 2.3 we see that

$$\limsup_{\substack{y\to x\\ \varepsilon\to 0}} \varepsilon \log P_y^\varepsilon[C] \leqq \limsup_{y\to x}\left[-\inf_{y_1,y_n\in A_y}\frac{1}{2}\sum\frac{d^2(y_{j-1},y_j)}{t_j-t_{j-1}}\right]$$

$$\leqq -\inf_{y_1,y_n\in A_x}\frac{1}{2}\sum\frac{d^2(y_{j-1},y_j)}{t_j-t_{j-1}}$$

$$= -\inf_{\omega\in C_x} I(\omega)\ .$$

The first step follows from straight-forward estimation of the integral (3.3) using the uniformity assured by Theorem 2.5. Since A is closed and bounded, points chosen in A_y have limit points in A_x as y approaches x. This justifies the second step. The last step is the identification furnished by Lemma 2.3.

If A is unbounded, we choose a Λ large enough so that

$$\Lambda > \inf_{\omega\in C_x} I(\omega)\ . \tag{3.4}$$

There is a standard estimate (Theorem 0.5, page 229, Volume II of [3]) of the form

$$p_\varepsilon(t,x,y) \leqq \frac{M}{(t\varepsilon)^{k/2}}\exp\left\{-\frac{\alpha}{2t\varepsilon}|x-y|^2\right\}. \tag{3.5}$$

We now define

$$S_\rho = \{|y_0| \leqq \rho, \cdots, |y_n| \leqq \rho\}$$

and S_ρ^* as the complement of S_ρ. From (3.5) we see that we can choose ρ large enough so that for all y in a compact set K

$$P_y^\varepsilon[T_\pi^{-1}S_\rho^*] \leqq Me^{-\Lambda/\varepsilon}\ . \tag{3.6}$$

Let us set

$$C_\rho = T_\pi^{-1}[A\cap S_\rho]\ ,$$

$$C_\rho^* = T_\pi^{-1}[S_\rho^*]\ .$$

Since $A\cap S_\rho$ is bounded and closed,

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon\log P_y^\varepsilon[C_\rho] \leqq -\inf_{\substack{\omega\in C\rho\\ \omega\in C_x}} I(\omega)$$

$$\leqq -\inf_{\omega\in C_x} I(\omega)\ .$$

From (3.6) we have

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon\log P_y^\varepsilon[C_\rho^*] \leqq -\Lambda\ .$$

Since $P_y^\varepsilon[C] \leqq P_y^\varepsilon[C_\rho] + P_y^\varepsilon[C_\rho^*]$,

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon\log P_y^\varepsilon[C] \leqq -\inf_{\omega\in C_x} I(\omega)\ .$$

We now choose for any integer n equally spaced time points $0 = t_0 < t_1 < \cdots t_n = T$. The partition has a length $h_n = T/n$. We denote by ω_π the trajectory obtained from ω by taking the points $\{\omega(t_j)\}$ and joining the successive ones by geodesics with natural parametrization. If there are several geodesics between two such points it is immaterial which one is chosen. In this way we obtain for each ω a ω_π which is piecewise geodesic and

$$\omega(t_j) = \omega_\pi(t_j)\ , \qquad j = 0, 1, 2, \cdots, n\ .$$

LEMMA 3.2. *For every* $\delta > 0$,

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\sup_x \varepsilon\log P_x^\varepsilon[\|\omega-\omega_\pi\| \geqq \delta] = -\infty\ .$$

Proof: In [7] it was proved (Theorem 3.5) that if $\phi(\lambda, z)$ is the solution to the equation

$$L\phi = \lambda\phi \qquad \text{for} \qquad z\in U\ ,$$

$$\phi = 1 \qquad \text{for} \qquad z\in \delta U\ ,$$

where U is the set

$$U = \{z : d(x,z) < \eta\}\ ,$$

with a boundary δU, then for every $\mu > 0$ there is a constant M_μ such that, for all $\eta \leqq \eta_0$ and for all $x\in R_k$,

$$\phi(\lambda, x) \leqq M_\mu \exp\{-\sqrt{2\lambda}\,\eta(1-\mu)\}\ . \tag{3.7}$$

Noting that if τ is the first exit time for the process starting at x for the set U, we then have

$$\phi(\lambda, x) = E_x[e^{-\lambda\tau}]\ .$$

We conclude that the expression

$$\phi(\varepsilon,\lambda,x) = \int e^{-\lambda\tau}\,dP_x^\varepsilon \tag{3.8}$$

satisfies the equation

$$\varepsilon L\phi = \lambda\phi \quad \text{in} \quad U,$$
$$\phi = 1 \quad \text{on} \quad \delta U.$$

Consequently, from (3.7), we have

$$\phi(\varepsilon, \lambda, x) \leqq M_\mu \exp\left\{-\sqrt{\frac{2\lambda}{\varepsilon}}\,\eta(1-\mu)\right\}.$$

Choosing $\mu = 1 - 1/\sqrt{2}$, we obtain

$$\phi(\varepsilon, \lambda, x) \leqq M \exp\left\{-\sqrt{\frac{\lambda}{\varepsilon}}\,\eta\right\}.$$

Using (3.8), a Tchebechev type of estimate yields

$$P_x^\varepsilon[\tau \leqq h] \leqq M \exp\left\{-\frac{\eta^2}{4\varepsilon h}\right\}.$$

We notice that

$$P_x^\varepsilon[\tau \leqq h] = P_x^\varepsilon\left[\sup_{0 \leqq t \leqq h} d(\omega(t), \omega(0)) \geqq \eta\right]$$

and we therefore obtain, for all x and $\eta \leqq \eta_0$,

$$P_x^\varepsilon\left[\sup_{0 \leqq t \leqq h} d(\omega(t), \omega(0)) \geqq \eta\right] \leqq M \exp\left\{-\frac{\eta^2}{4\varepsilon h}\right\}. \tag{3.9}$$

We want to estimate

$$P_x^\varepsilon\left[\sup_{0 \leqq t \leqq T} d(\omega(t), \omega_\pi(t)) \geqq 2\eta\right].$$

Since $\omega(t_j) = \omega_\pi(t_j)$ for $j = 0, 1, \cdots, n$, and ω_π is piecewise geodesic for t in (t_j, t_{j+1}),

$$d(\omega_\pi(t), \omega_\pi(t_j)) \leqq d(\omega_\pi(t_{j+1}), \omega_\pi(t_j)) = d(\omega(t_{j+1}), \omega(t_j))$$
$$\leqq \sup_{t_j \leqq t \leqq t_{j+1}} d(\omega(t_j), \omega(t)).$$

We now have

$$\sup_{0 \leqq t \leqq T} d(\omega(t), \omega_\pi(t)) = \sup_{0 \leqq j \leqq n-1}\ \sup_{t_j \leqq t \leqq t_{j+1}} d(\omega(t), \omega_\pi(t))$$
$$\leqq \sup_{0 \leqq j \leqq n-1}\ \sup_{t_j \leqq t \leqq t_{j+1}} [d(\omega(t), \omega(t_j)) + d(\omega_\pi(t), \omega_\pi(t_j))] \tag{3.10}$$
$$\leqq 2 \sup_{0 \leqq j \leqq n-1}\ \sup_{t_j \leqq t \leqq t_{j+1}} d(\omega(t), \omega(t_j)).$$

From the Markov property and the inequality (3.10), we obtain

$$P_x^\varepsilon\left[\sup_{0 \leqq t \leqq T} d(\omega(t), \omega_\pi(t)) \geqq 2\eta\right] \leqq P_x^\varepsilon\left[\sup_{0 \leqq j \leqq n-1}\ \sup_{t_j \leqq t \leqq t_{j+1}} d(\omega(t), \omega(t_j)) \geqq \eta\right]$$
$$\leqq \sum_{j=0}^{n-1} P_x^\varepsilon\left[\sup_{t_j \leqq t \leqq t_{j+1}} d(\omega(t), \omega(t_j)) \geqq \eta\right]$$
$$\leqq n \sup_x P_x^\varepsilon\left[\sup_{0 \leqq t \leqq h} d(\omega(t), \omega(0)) \geqq \eta\right]$$
$$\leqq nM \exp\left\{-\frac{\eta^2}{4\varepsilon h}\right\}.$$

Since $|x - y| \leqq \beta\, d(x, y)$ for some β, we have for $\delta \leqq \delta_0$

$$P_x^\varepsilon[\|\omega - \omega_\pi\| \geqq \delta] \leqq nM \exp\left\{-\frac{\delta^2}{16\beta^2 \varepsilon h}\right\}.$$

Consequently,

$$\limsup_{n \to \infty}\ \limsup_{\varepsilon \to 0}\ \sup_x\ \varepsilon \log P_x^\varepsilon[\|\omega - \omega_\pi\| \geqq \delta] = -\infty.$$

THEOREM 3.3. *For any closed set $C \subset \Omega$,*

$$\limsup_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log P_y^\varepsilon[C] \leqq -\inf_{\omega \in C_x} I(\omega).$$

Proof: Let us define for $\delta > 0$ the sets C_δ and C_δ^δ as

$$C_\delta = \{\omega : \omega \in C\ |\omega(0) - x| < \delta\},$$
$$C_\delta^\delta = \{z : \|z - \omega\| < \delta \text{ for some } \omega \in C_\delta\}.$$

We introduce the function

$$I^\delta(\omega) = \inf_{z : \|z - \omega\| < \delta} I(z).$$

Then $I^\delta(\omega)$ is finite everywhere. Moreover if

$$T_\delta = \inf_{\omega \in C_\delta^\delta} I(\omega),$$

then $\omega \in C_\delta \Rightarrow I^\delta(\omega) \geqq T_\delta$. Since $y \to x$ we can assume $|y - x| < \delta$. We would therefore obtain

$$P_y^\varepsilon[C] = P_y^\varepsilon[C_\delta] \leqq P_y^\varepsilon[I^\delta(\omega) \geqq T_\delta].$$

Further,

$$P_y^\varepsilon[I^\delta(\omega) \geqq T_\delta] \leqq P_y^\varepsilon[\|\omega - \omega_\pi\| \geqq \delta] + P_y^\varepsilon[I(\omega_\pi) \geqq T_\delta] \tag{3.11}$$

and

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \sup_x \varepsilon \log P_y^\varepsilon[\|\omega - \omega_\pi\| \geqq \delta] = -\infty . \tag{3.12}$$

The set $[I(\omega_\pi) \geqq T_\delta]$ is equal to

$$\left\{\omega : \frac{1}{2}\sum_{j=0}^{n-1} \frac{d^2(\omega(t_j), \omega(t_{j+1}))}{t_{j+1} - t_j} \geqq T_\delta\right\} .$$

Hence for every n,

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[I(\omega_\pi) \geqq T_\delta] \leqq -T_\delta . \tag{3.13}$$

It follows from (3.11), (3.12) and (3.13) that

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C] \leqq -T_\delta .$$

This is true for every $\delta > 0$ and $\lim_{\delta\to 0} T_\delta = T$, where

$$T = \inf_{\omega \in C_x} I(\omega) .$$

The proof of the theorem is now complete.

Lemma 3.4. *Let $f \in \Omega$ and $f(0) = x$. Let V be the set $V = \{\omega : \|\omega - f\| < \delta\}$, where $\delta > 0$. Then*

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[U] \geqq -I(f) .$$

Proof: We can assume without loss of generality that δ is small and that $I(f) = a < \infty$. Let V_π denote the set

$$V_\pi = \{\omega : |\omega(t_j) - f(t_j)| \leqq \tfrac{1}{2}\delta \text{ for } j = 0, 1, 2, \cdots, n\} ,$$

then

$$P_y^\varepsilon[V_\pi] = \int_{A_\pi} \prod p_\varepsilon(t_j - t_{j-1}, y_{j-1}, y_j)\, dy_j ,$$

where A_π is the set

$$A_\pi = \{y : |y_j - f(t_j)| \leqq \tfrac{1}{2}\delta \text{ for } j = 0, 1, \cdots, n\} .$$

Using Theorem 2.3, one can obtain

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[V_\pi] \geqq -I(f_\pi) \geqq -I(f) .$$

It is therefore sufficient to prove that

$$\lim_{n\to\infty} \limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[U_\pi - U] = -\infty . \tag{3.14}$$

We see that $\omega \in V_\pi - V \Rightarrow |\omega(t_j) - f(t_j)| \leqq \frac{1}{2}\delta$ for all j but $\|\omega(t) - f(t)\| \geqq \delta$. This implies that, for some interval $t_j \leqq t \leqq t_{j+1}$,

$$|\omega(t_j) - f(t_j)| < \tfrac{1}{2}\delta , \qquad |\omega(t_{j+1}) - f(t_{j+1})| < \tfrac{1}{2}\delta ,$$

but $|\omega(\tau) - f(\tau)| \geqq \delta$ for some τ in $t_j \leqq \tau \leqq t_{j+1}$. Further, we can choose n large enough so that, for the resulting π, $|f(\tau) - f(t_j)| < \frac{1}{4}\delta$ for all τ in $t_j \leqq \tau \leqq t_{j+1}$. We conclude from all this that, if $\omega \in V_\pi - V$, then $|\omega(\tau) - \omega(t_j)| > \frac{1}{4}\delta$ for τ in (t_j, t_{j+1}) for some j. Hence $d(\omega(\tau), \omega(t_j)) \geqq \beta\frac{1}{4}\delta$ for some $\beta > 0$. This in turn implies that $I(\omega) \geqq \beta^2\delta^2/32h$. Moreover, $V_\pi - V$ is closed and therefore, by Theorem 3.3,

$$\lim_{n\to\infty} \limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[V_\pi - V] = -\infty .$$

This establishes (3.14) and hence the lemma is proved.

Theorem 3.5. *Let $G \subset \Omega$ be open. Then*

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[G] \geqq -\inf_{\omega \in G_x} I(\omega) .$$

Proof: Since $G \subset \Omega$ is open for $f \in \Omega$, there is a sphere V around f contained in Ω. By Lemma 3.4,

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[G] \geqq -I(f) .$$

This is true for all $f \in \Omega$ with $f(0) = x$. Hence the theorem is true. Theorems 3.3 and 3.5 contain Theorem 2.4.

We conclude this section with a corollary to Theorem 3.5.

Corollary 3.6. *Let $A \subset \Omega$ be measurable, A° and $\bar{A}$ the interior and closure of A. Assume that for $x \in K$ (K compact)*

$$\phi(x) = \inf_{\omega \in \bar{A} \cap \Omega_x} I(\omega) = \inf_{\omega \in A^\circ \cap \Omega_x} I(\omega) .$$

Then $\phi(x)$ is continuous on K and

$$\lim_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon[A] = -\phi(x)$$

uniformly over K.

4. Fundamental Solution in a Subdomain

$D \subset R_k$ is an open set and we kill the process at the boundary B of D. We consider the fundamental solution $q_\varepsilon(t, x, y)$ of the equation

$$\frac{\partial q}{\partial t} = \tfrac{1}{2}\varepsilon \sum a_{ij}(x) \frac{\partial^2 q}{\partial x_i \, \partial x_j} \qquad \text{for} \qquad x \in D, \tag{4.1}$$

$$q_\varepsilon(t, x, y) \to 0 \qquad \text{as} \qquad x \to B .$$

It is clear that

$$q_\varepsilon(t, x, y) \leqq p_\varepsilon(t, x, y) .$$

Moreover, if we define the set function

$$q_\varepsilon(t, x, A) = \int_A q_\varepsilon(t, x, y)\, dy \tag{4.2}$$

for sets $A \subset D$, then the connection between (4.2) and the measures P_x^ε is

$$q_\varepsilon(t, x, A) = P_x^\varepsilon\{\omega : \omega(t) \in A,\ \omega(\tau) \in D \text{ for } 0 \leqq \tau \leqq t\} . \tag{4.3}$$

Let us introduce two subsets of Ω,

$$G = \{\omega : \omega(\tau) \in D \text{ for } 0 \leqq \tau \leqq T\},$$

and

$$\bar{G} = \text{closure of } G .$$

Then $\bar{G} \subset \{\omega : \omega(\tau) \in D \cup B \text{ for } 0 \leqq \tau \leqq T\}$. In some cases the inclusion is proper and that is relevant. We introduce the distances

$$\underline{d}_D(x, y) = \inf_{\substack{\omega(0)=x\\ \omega(T)=y\\ \omega\in\bar{G}}} l(\omega) ,$$

and

$$\bar{d}_D(x, y) = \inf_{\substack{\omega(0)=x\\ \omega(T)=y\\ \omega\in G}} l(\omega) .$$

The set D is called "smooth" if $\underline{d}_D(x, y) = \bar{d}_D(x, y)$. This means that if a trajectory of length l can be approximated by trajectories in D, then it can be also approximated by trajectories within D whose lengths approach l. In such a case we shall denote by $d_D(x, y)$ the common value of $\underline{d}_D$ and $\bar{d}_D$. We can check that the following relations hold:

$$\inf_{\substack{\omega\in G\\ \omega(0)=x\\ \omega(T)=y}} I(\omega) = \frac{1}{2T} \bar{d}_D^2(x, y) ,$$

$$\inf_{\substack{\omega\in \bar{G}\\ \omega(0)=x\\ \omega(T)=y}} I(\omega) = \frac{1}{2T} \underline{d}_D^2(x, y) . \tag{4.4}$$

LEMMA 4.1. *For $A \subset D$ and closed,*

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log q_\varepsilon(T, y, A) \leqq -\inf_{z\in A} \frac{\underline{d}_D^2(x, z)}{2T} .$$

Proof: Let us take $C \subset \Omega$ as

$$C = \{\omega : \omega(T) \in A,\ \omega \in \bar{G}\} .$$

Then from relation (4.3), we have

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log q_\varepsilon(T, y, A) \leqq \limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C]$$

$$\leqq -\inf_{\omega\in C_x} I(\omega) = -\frac{1}{2T} \inf_{z\in A} \underline{d}_D^2(x, z) .$$

Lemma 4.1 is only a weak version of what we want to prove. To study the density $q_\varepsilon(t, x, y)$ we use a slightly different approach.

LEMMA 4.2. *Let x, y be points in D. Then*

$$\limsup_{\substack{\varepsilon\to 0\\ x'\to x\\ y'\to y}} \varepsilon \log q_\varepsilon(T, y, y') \leqq -\frac{1}{2T} \underline{d}_D^2(x, y) .$$

Proof: Let us take a neighborhood S_δ around y of radius δ and a neighborhood $S_{2\delta}$ around S_δ of radius δ. Let τ be the first time the set $S_{2\delta}$ is reached starting from x'. Then for $y' \in S_\delta$,

$$q_\varepsilon(T, x', y') = \int_{\substack{\tau < T\\ \omega\in G}} q_\varepsilon(t - \tau, \omega(\tau), y') P_{x'}^\varepsilon(d\omega) . \tag{4.5}$$

Since $\omega(\tau)$ is in the boundary of $S_{2\delta}$ and y' is in S_δ, we have $|\omega(\tau) - y'| \geqq \delta$. Moreover, $q_\varepsilon(t, x, y) \leqq p_\varepsilon(t, x, y)$ and, if x and y are kept away, the functions are bounded. Hence for all $\omega(\tau)$ and y',

$$q_\varepsilon(T - \tau, \omega(\tau), y') \leqq M_\delta < \infty .$$

Continuing with (4.5) this yields

$$q_\varepsilon(T, x', y') \leqq M_\delta P^\varepsilon_{x'}[\omega : \omega \in G, \tau \leqq T] .$$

Therefore,

$$\limsup_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log q_\varepsilon(T, x', y') \leqq -\frac{1}{2T} \inf_{z \in S_{2\delta}} \bar{d}^2_D(x, z) .$$

Since this is true for every $\delta > 0$, we prove the lemma by letting $\delta \to 0$.

LEMMA 4.3. *For $A \subset D$ and open,*

$$\liminf_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log q_\varepsilon(T, y, A) \geqq -\inf_{z \in A} \frac{\bar{d}^2_D(x, z)}{2T} .$$

Proof: We have

$$q_\varepsilon(T, y, A) = P^\varepsilon_y[\omega : \omega \in G, \omega(T) \in A] .$$

The set on the right is open. A direct application of Theorem 3.5 yields

$$\liminf_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log q_\varepsilon(T, y, A) \geqq -\inf_{\substack{\omega \in G_x \\ \omega(T) \in A}} I(\omega)$$

$$= -\frac{1}{2T} \inf_{z \in A} \bar{d}^2_D(x, z) .$$

LEMMA 4.4. *Let $K \subset D$ be compact. Then for $\Delta > 0$,*

$$\liminf_{\substack{\varepsilon \to 0 \\ x, y \in K \\ |x-y| \to 0}} \varepsilon \log q_\varepsilon(\Delta, x, y) \geqq 0 .$$

Proof: We use the formula

$$q_\varepsilon(\Delta, x, y) = p_\varepsilon(\Delta, x, y) - \int_{\tau \leqq \Delta} p_\varepsilon(\Delta - \tau, \omega(\tau), y) P^\varepsilon_x(d\omega) ,$$

where τ is the first exit time from D. Since $y \in K \subset D$ and K is compact, $|y - \omega(\tau)| \geqq \eta > 0$ for all $y \in K$ and $\omega(\tau) \in B$. Therefore,

$$q_\varepsilon(\Delta, x, y) \geqq p_\varepsilon(\Delta, x, y) - \sup_{\substack{b \in K \\ 0 \leqq u \leqq \Delta \\ y \in B}} p_\varepsilon(u, b, y)$$

(4.6)

$$\geqq p_\varepsilon(\Delta, x, y) - Me^{-c\eta^2/\varepsilon} .$$

Since by Theorem 2.5

$$\liminf_{\substack{\varepsilon \to 0 \\ |x-y| \to 0}} \varepsilon \log p_\varepsilon(\Delta, x, y) \geqq 0 ,$$

it follows from (4.6) that

$$\liminf_{\substack{\varepsilon \to 0 \\ x, y \in K \\ |x-y| \to 0}} \varepsilon \log q_\varepsilon(\Delta, x, y) \geqq 0 .$$

LEMMA 4.5. *Let x, y be points in D. Then*

$$\liminf_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log q_\varepsilon(T, x', y') \geqq -\frac{1}{2T} \bar{d}^2_D(x, y) .$$

Proof: Since $q_\varepsilon(t, x, y)$ satisfies the semigroup property,

$$q_\varepsilon(T, x', y') = \int_D q_\varepsilon(T - \Delta, x', z) q_\varepsilon(\Delta, z, y') \, dz$$

$$\geqq \int_{z \in S} q_\varepsilon(T - \Delta, x', z) q_\varepsilon(\Delta, z, y') \, dz$$

$$\geqq q_\varepsilon(T - \Delta, x', S) \inf_{z \in S} q_\varepsilon(\Delta, z, y') .$$

We take S to be a small sphere around y of radius δ. Then,

$$\liminf_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log q_\varepsilon(T : x', y') \geqq \liminf_{\substack{\varepsilon \to 0 \\ x' \to x}} \varepsilon \log q_\varepsilon(T - \Delta, x', S)$$

$$+ \liminf_{\substack{\varepsilon \to 0 \\ y' \to y}} \inf_{z \in S} \varepsilon \log q_\varepsilon(\Delta, z, y') .$$

Combining Lemmas 4.3 and 4.4, we have if we let $\delta \to 0$

$$\liminf_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log q_\varepsilon(T, x', y') \geqq -\frac{1}{2(T - \Delta)} \bar{d}^2_D(x, y) .$$

Since $\Delta > 0$ is arbitrary, we complete the proof of Lemma 4.5 by letting $\Delta \to 0$.

Lemmas 4.2 and 4.5 yield

THEOREM 4.6. *Let D be smooth. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \log q_\varepsilon(T, x, y) = -\frac{1}{2T} d_D^2(x, y)$$

uniformly for $x, y \in K$, where $K \subset D$ is any compact set.

COROLLARY 4.7. *For D smooth,*

$$\lim_{t \to 0} t \log q(t, x, y) = -\tfrac{1}{2} d_D^2(x, y) .$$

Let us set

$$R_\varepsilon(t, x, y) = p_\varepsilon(t, x, y) - q_\varepsilon(t, x, y) , \tag{4.7}$$

$$d_D^*(x, y) = \inf_{\substack{\omega(0)=x \\ \omega(T)=y \\ \omega \notin G}} l(\omega) . \tag{4.8}$$

Then

$$\inf_{\substack{\omega(0)=x \\ \omega(T)=y \\ \omega \notin G}} I(\omega) = \frac{1}{2T} d_D^{*2}(x, y) .$$

Moreover, it is easily verified that

$$d_D^*(x, y) = \inf_{z \in B} [d(x, z) + d(z, y)] .$$

LEMMA 4.8.

$$\limsup_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log R_\varepsilon(T, x', y') \leqq -\frac{1}{2T} d_D^{*2}(x, y) .$$

Proof: This is proved in a manner similar to Lemma 4.2. In order to get to y' from x' by a trajectory not in G, we have to reach the boundary first and then come to a sphere around y. From this point on the proof is exactly like that of Lemma 4.2. Since τ is defined differently, we end up with

$$R_\varepsilon(T, x', y') = \int_{\tau \leqq T} p_\varepsilon(t - \tau, \omega(\tau), y') P_{x'}^\varepsilon(d\omega) ,$$

where τ is the first time the trajectory enters a set $S_{2\delta}$ around y after reaching the boundary B of D. Hence, by suitably letting $\delta \to 0$, we have

$$\limsup_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log R_\varepsilon(T, x', y') \leqq -\frac{1}{2T} d_D^*(x, y) .$$

For any region D with boundary B, Lemmas 4.8, 4.2 and Theorem 2.5 imply

$$\begin{aligned} &\lim_{t \to 0} 2t \log p(t, x, y) = -d^2(x, y) , \\ &\limsup_{t \to 0} 2t \log q(t, x, y) \leqq d_D^2(x, y) , \\ &\limsup_{t \to 0} 2t \log R(t, x, y) \leqq -d_D^{*2}(x, y) . \end{aligned} \tag{4.9}$$

An immediate consequence of (4.9) is

THEOREM 4.9. *Let x, y in D be such that all the geodesics from x, y lie completely within D. Then the same is true for all a, b near x, y, respectively, and*

$$\lim_{t \to 0} \frac{q(t, a, b)}{p(t, a, b)} = 1$$

uniformly for a, b near x, y. On the other hand, let x, y in D be such that no geodesic from x to y can be approximated uniformly by paths completely in D. Then the same is true for all a, b near x, y, respectively, and

$$\lim_{t \to 0} \frac{q(t, a, b)}{p(t, a, b)} = 0$$

uniformly for a, b near x, y.

5. Some Associated Differential Equations

Let $D \subset R_k$ be a domain with boundary B. We assume throughout this section that the set of points $b \in B$, with the property that every sufficiently small neighborhood N_b around b is disconnected by B, is dense in B.

Let $\phi(x, \lambda)$ be the solution of the equation

$$\begin{aligned} \tfrac{1}{2} \sum a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \, \partial x_j} &= \lambda \phi \quad \text{for} \quad x \in D , \\ \phi &= 1 \quad \text{on} \quad B . \end{aligned} \tag{5.1}$$

We then have

THEOREM 5.1.

$$\lim_{\lambda\to\infty} -\frac{1}{\sqrt{2\lambda}} \log \phi(x, \lambda) = d(x, B)$$

uniformly in x over compact subsets of $D \cup B$.

Proof: Let τ be the exit time from the set D. Then the solution $\phi(x, \lambda)$ of equation (5.1) has a representation

$$\phi(x, \lambda) = \int e^{-\lambda\tau} P_x(d\omega)$$

$$= \int e^{-\lambda t} F(x, dt) ,$$

where

$$F(x, t) = P_x[\tau \leqq t] .$$

In order to estimate $\phi(x, \lambda)$ for large λ we need estimate $F(x, t)$ for small t:

$$F(x, \varepsilon) = P_x[\tau \leqq \varepsilon] = P_x^\varepsilon[\tau \leqq 1] .$$

Let C be the set

$$C = \{\omega : \omega(s) \in B \text{ for some } s, \quad 0 \leqq s \leqq 1\}.$$

Then C is closed and $C = \{\omega : \tau \leqq 1\}$. Therefore,

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log F(y, \varepsilon) = \limsup_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C] \tag{5.2}$$

$$\leqq -\inf_{C_x} I(\omega) = -\tfrac{1}{2}d^2(x, B) .$$

Let G be defined as the set of trajectories ω for which there are two points $0 \leqq u_1 < u_2 \leqq 1$ in the time interval $0 \leqq s \leqq 1$ such that, in the interval $u_1 \leqq \tau \leqq u_2$, $\omega(\tau)$ lies completely in one of the disconnected neighborhoods N_b and $\omega(u_1)$ and $\omega(u_2)$ belong to two different components of the disconnected set N_b. It is clear that G is open and $G \subset C$. Therefore,

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[C] \geqq \liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log P_y^\varepsilon[G] \geqq -\inf_{\omega\in G_x} I(\omega) .$$

Let $b \in B$ and N_b be a small neighborhood around b which is disconnected. Consider the trajectory $\omega(\tau)$, $0 \leqq \tau \leqq 1$, which is a geodesic from x to b, to be naturally parametrized. Then $I(\omega) = \frac{1}{2}d^2(x, b)$. The trajectory $\omega(\tau)$ can be extended slightly so that it penetrates into two different components of N_b and lies completely in N_b in between. This can always be achieved by choosing N_b small with only a slight increase in $I(\omega)$. The parametrization is to be adjusted by rescaling time so that the time interval is still $0 \leqq s \leqq 1$. The final conclusion is

$$\inf_{\omega\in G_x} I(\omega) \leqq \inf_{b\in B_0} \tfrac{1}{2}d^2(x, b) ,$$

where B_0 is the set of points in B for which the construction is possible. By assumption, B_0 is dense in B. Consequently,

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log F(y, \varepsilon) \geqq -\tfrac{1}{2}d^2(x, B) .$$

Along with (5.2) this implies

$$\lim_{\substack{\varepsilon\to 0\\ y\to x}} \varepsilon \log F(y, \varepsilon) = -\tfrac{1}{2}d^2(x, B) ,$$

since

$$\phi(x, \lambda) = \int e^{-\lambda t} F(x, dt) .$$

Standard estimation yields

$$\lim_{\substack{\lambda\to\infty\\ y\to x}} \frac{1}{\sqrt{2\lambda}} \log \phi(y, \lambda) = -d(x, B) .$$

We still assume that we have a domain D with a boundary B satisfying the condition described early in the section. Let us define for any set $\Gamma \subset B$ (Γ measurable) the function $\psi(x, \lambda, \Gamma)$ as the solution of the equation

$$\begin{aligned} L\psi &= \lambda\psi && \text{for} \quad x \in D , \\ \psi &= 1 && \text{for} \quad x \in \Gamma , \\ \psi &= 0 && \text{for} \quad x \in B - \Gamma . \end{aligned} \tag{5.3}$$

Then $\psi(x, \lambda, \Gamma)$ has a representation

$$\psi(x, \lambda, \Gamma) = \int e^{-\lambda t} F(x, \Gamma, dt) . \tag{5.4}$$

If τ is the first exit time from D, then

$$F(x, \Gamma, t) = P_x[\tau \leqq t, \omega(\tau) \in \Gamma] . \tag{5.5}$$

We assume that D is smooth in the sense of Section 4. Moreover, we assume that the property extends to the boundary B. The actual assumptions are as follows: If

$$G = \{\omega : \omega(\tau) \in D \quad \text{for} \quad 0 \leqq \tau < T\},$$

$$\bar{G} = \text{closure of } G,$$

$$\bar{d}_D(x, y) = \inf_{\substack{\omega \in G \\ \omega(0)=x \\ \omega(T)=y}} l(\omega),$$

$$\underline{d}_D(x, y) = \inf_{\substack{\omega \in \bar{G} \\ \omega(0)=x \\ \omega(T)=y}} l(\omega),$$

then $\bar{d}_D(x, y) = \underline{d}_D(x, y)$ for all $x \in D$ and $y \in D \cup B$. We denote the common value by $d_D(x, y)$ and by $d_D(x, b)$ if b is on the boundary.

Lemma 5.2. *For $\Gamma \subset B$ open and $x \in D$,*

$$\liminf_{\substack{\lambda \to \infty \\ y \to x}} \frac{1}{\sqrt{2\lambda}} \log \psi(y, \lambda, \Gamma) \geqq -\inf_{b \in \Gamma} d_D(x, b).$$

For $\Gamma \subset B$ closed and $x \in D$,

$$\limsup_{\substack{\lambda \to \infty \\ y \to x}} \frac{1}{\sqrt{2\lambda}} \psi(y, \lambda, \Gamma) \leqq -\inf_{b \in \Gamma} d_D(x, b).$$

Proof: As in Theorem 5.1 we study

$$F(x, \Gamma, \varepsilon) = P_x[\tau \leqq \varepsilon, \omega(\tau) \in \Gamma] = P_x^\varepsilon[\tau \leqq 1, \omega(\tau) \in \Gamma].$$

Let U be the set

$$U = \{\omega : \tau \leqq 1, \omega(\tau) \in \Gamma\}.$$

In order to show that

$$\liminf_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log F(y, \Gamma, \varepsilon) \geqq -\tfrac{1}{2} \inf_{b \in \Gamma} d_D^2(x, b), \tag{5.6}$$

it suffices to show that for any given $b \in \Gamma$ and $\delta > 0$ there exists an interior point ω_0 of U (U is not open in general) such that

$$I(\omega_0) \leqq \tfrac{1}{2} d_D^2(x, b) + \delta. \tag{5.7}$$

Let $b \in \Gamma$ be given. Then there is a b' near b for which the disconnectedness condition holds. Let us take a ω' in G with $\omega'(1) = b$ and $I(\omega')$ nearly equal to $\frac{1}{2} d_D^2(x, b)$. We take a point near b on ω' inside D and connect the point with b' by a geodesic not necessarily inside D. Then we repeat the construction of Theorem 5.1 to get a ω_0 which penetrates into two different components of a small disconnected neighborhood $N_{b'}$ around b'. Any trajectory close to ω_0 will have to exit at a point near b or b'. Since Γ is open, things can be managed so that all these trajectories exit in Γ. $I(\omega_0)$ of course is not much larger than $\frac{1}{2} d_D^2(x, b)$. This proves (5.7) and hence (5.6). Now let Γ be closed. Then the set

$$C = \{\omega : \omega(s) \in \Gamma \text{ for some } 0 \leqq s \leqq 1, \omega \in \bar{G}\}$$

is closed and obviously

$$\begin{aligned} \limsup_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log F(y, \Gamma, \varepsilon) &\leqq \limsup_{\substack{\varepsilon \to 0 \\ y \to x}} \varepsilon \log P_y^\varepsilon[C] \\ &\leqq -\inf_{\omega \in C_x} I(\omega) = -\tfrac{1}{2} \inf_{b \in \Gamma} d_D^2(x, b). \end{aligned} \tag{5.8}$$

The lemma is then proved using the relation (5.4) and the properties (5.6) and (5.8).

We now consider the solution $\phi(x, \lambda)$ to the equation

$$\begin{aligned} L\phi &= \lambda\phi \quad &\text{for} \quad x \in D, \\ \phi &= h(b, \lambda) \quad &\text{for} \quad b \in B, \end{aligned} \tag{5.9}$$

where

$$h(b, \lambda) = \exp\{\sqrt{2\lambda}\, h(b)\} \tag{5.10}$$

and $h(b)$ is a bounded continuous function on B.

Theorem 5.3.

$$\lim_{\lambda \to \infty} \frac{1}{\sqrt{2\lambda}} \log \phi(x, \lambda) = \sup_{b \in B} [h(b) - d_D(x, b)]$$

uniformly over compact subsets $K \subset D$.

Proof: We use the formula

$$\phi(x, \lambda) = \int h(b, \lambda) \psi(x, \lambda, db).$$

Lemma 5.2 is all we need to be able to use Theorem 3.4 of [6] and the theorem follows. It is elementary to see that the uniformity in Lemma 5.2 carries over to the uniformity asserted in the theorem.

Finally we turn to the following initial value problem: $u_\varepsilon(t, x)$ is the solution of the equation

$$\frac{\partial u_\varepsilon}{\partial t} = \tfrac{1}{2}\varepsilon L u_\varepsilon + \tfrac{1}{2}\sum a_{ij}(x)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j} + P(t, x), \tag{5.11}$$
$$u_\varepsilon(0, x) = V(x).$$

Let us suppose for simplicity that $V(x)$ and $P(t, x)$ are continuous and bounded. We are interested in the behavior of $u_\varepsilon(t, x)$ as $\varepsilon \to 0$.

THEOREM 5.4.

$$\lim_{\varepsilon\to 0} u_\varepsilon(t, x) = u(t, x)$$

exists and

$$u(t, x) = \sup_{\omega\in\Omega_x}\left[V(\omega(t)) + \int_0^t P(t-\tau, \omega-\tau)\, d\tau - I(\omega)\right].$$

Proof: Let us set

$$v_\varepsilon(t, x) = \exp\left\{\frac{1}{\varepsilon}u_\varepsilon(t, x)\right\}.$$

Then the equation satisfied by v_ε is

$$\frac{\partial v_\varepsilon}{\partial t} = \tfrac{1}{2}\varepsilon L v_\varepsilon + \frac{1}{\varepsilon}v_\varepsilon P, \tag{5.12}$$
$$v_\varepsilon(0, x) = \exp\left\{\frac{1}{\varepsilon}V(x)\right\};$$

hence the solution is given by the function space integral

$$v_\varepsilon(t, x) = \int \exp\left\{\frac{1}{\varepsilon}V(\omega(t)) + \frac{1}{\varepsilon}\int_0^t P(t-\tau, \omega(\tau))\, d\tau\right\}P_x^\varepsilon(d\omega).$$

We now appeal to Theorem 3.4 in [5] and conclude that

$$\lim_{\varepsilon\to 0} u_\varepsilon(t, x) = \lim_{\varepsilon\to 0}\varepsilon\log v_\varepsilon(t, x)$$
$$= \sup_{\omega\in\Omega_x}\left[V(\omega(t)) + \int_0^t P(t-\tau, \omega(\tau))\, d\tau - I(\omega)\right].$$

Remark. With a little more effort one can show that the limit in Theorem 5.4 is actually uniform over compact subsets of t and x. One can also replace the boundedness of P and V by reasonable growth conditions. A similar approach for the case when L is the Laplacian was carried out in [2] using the results in [4].

6. The Case With a First Order Term

We assume that the operator (2.1) has an additional first order term:

$$L_0 f = \tfrac{1}{2}\sum a_{ij}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j} + \sum b_j(x)\frac{\partial f}{\partial x_j}.$$

The coefficients $\{a_{ij}(x)\}$ satisfy the assumptions postulated earlier. In addition we assume that

$$|b_j(x)| \leqq M < \infty,$$
$$|b_j(x) - b_j(y)| \leqq M|x-y|^\alpha.$$

Let $q(t, x, y)$ be the fundamental solution for the equation

$$\frac{\partial q}{\partial t} = L_0 q.$$

If we replace L_0 by εL_0, we get a family

$$q_\varepsilon(t, x, y) = q(\varepsilon t, x, y).$$

Using the densities $q_\varepsilon(t, x, y)$ we obtain a family of measures Q_x^ε. The following Theorem is proved in [5].

THEOREM 6.1. *For every x and ε the measure Q_x^ε on Ω is absolutely continuous with respect to P_x^ε and*

$$\frac{dQ_x^\varepsilon}{dP_x^\varepsilon} = \exp\left\{\int_0^T b(\omega(\tau))a^{-1}(\omega(\tau))\, d\omega(\tau) - \tfrac{1}{2}\varepsilon\int_0^T [b(\omega(\tau))a^{-1}(\omega(\tau))b(\omega(\tau))]\, d\tau\right\}.$$

Remark. Although the assumptions in [5] are stronger (the Hölder exponent α is assumed there to be 1) the theorem can be proved under less restrictive conditions. For instance in [3], Volume II, Chapter XIV, Section 6, it is carried out by a different method when $a_{ij}(x) = \delta_{ij}$. The same method will work in the present case. It is elementary to obtain the following estimate.

LEMMA 6.2. *If we denote by $F(\varepsilon, \omega)$ the exponent in the Radon-Nikodym derivative of Theorem 6.1, then*

$$\int \exp\{\lambda F(\varepsilon, \omega)\}P_x^\varepsilon(d\omega) \leqq \exp\{C\varepsilon\tfrac{1}{2}\lambda^2\}.$$

Proof: One uses the fact that

$$\exp\left\{\lambda\int_0^t b(\omega(\tau))a^{-1}(\omega(\tau))\, d\omega(\tau) - \tfrac{1}{2}\lambda^2\varepsilon\int_0^t [b(\omega(\tau))a^{-1}(\omega(\tau))b(\omega(\tau))]\, d\tau\right\}$$

is a martingale relative to P_x^ε.

THEOREM 6.3. *For $G \subset \Omega$ open and $C \subset \Omega$ closed,*

$$\liminf_{\substack{\varepsilon\to 0\\ y\to x}} Q_y^\varepsilon[G] \geqq -\inf_{\omega\in G_x} I(\omega)\,,$$

$$\limsup_{\substack{\varepsilon\to 0\\ y\to x}} Q_y^\varepsilon[C] \leqq -\inf_{\omega\in C_x} I(\omega)\,.$$

Proof: We have

$$Q_y^\varepsilon[G] = \int_G \exp\{F(\varepsilon,\omega)\}P_y^\varepsilon[d\omega]$$
$$\geqq e^{-A}P_y^\varepsilon[G\cap\{F(\varepsilon,\omega)\geqq -A\}]$$
$$\geqq e^{-A}P_y^\varepsilon[G] - e^{-A}P_y^\varepsilon[|F(\varepsilon,\omega)|\geqq A]\,.$$

It therefore suffices to show that

(6.1) $$\lim_{A\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P_y^\varepsilon[|F(\varepsilon,\omega)|\geqq A] = -\infty$$

uniformly in y. Similarly,

$$Q_y^\varepsilon[C] = \int_C \exp\{F(\varepsilon,\omega)\}P_y^\varepsilon(d\omega)$$
$$\leqq e^{A}P_y^\varepsilon[C] + \int_{[F(\varepsilon,\omega)\geqq A]}\exp\{F(\varepsilon,\omega)\}P_y^\varepsilon(d\omega)\,.$$

Hence we need only show that

(6.2) $$\lim_{A\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\int_{[F\geqq A]}\exp\{F(\varepsilon,\omega)\}P_y^\varepsilon(d\omega) = -\infty$$

uniformly in y. Relations (6.1) and (6.2) follow easily from Lemma 6.2.

THEOREM 6.4.

$$\lim_{t\to 0} -2t\log q(t,x,y) = d^2(x,y)$$

uniformly in x and y over compact sets.

Proof: We have the following estimates from Theorem 0.5, page 229, Volume II of [3]:

(6.3) $$q(t,x,y) \leqq Mt^{-k/2}\exp\left\{-\frac{\alpha}{2t}|x-y|^2\right\},$$

(6.4) $$q(t,x,y) \geqq M_1t^{-k/2}\exp\left\{-\frac{\alpha_1}{2t}|x-y|^2\right\} - M_2t^{-k/2+\lambda}\exp\left\{-\frac{\alpha_2}{2t}|x-y|^2\right\}.$$

UPPER BOUND. Let x, y be two points in R_k. We choose a sphere S_δ around y of radius δ and a sphere $S_{2\delta}$ of radius δ around $S_{2\delta}$. Let B denote the boundary of $S_{2\delta}$. We assume that $y'\to y$ and hence $y'\in S_\delta$. Let τ be the first entrance time into the set B for a process starting at x'. Then,

$$q(t,x',y') = \int_{\tau\leqq t} q(t-\tau,\omega(\tau),y')P_{x'}(d\omega)$$
$$\leqq \sup_{\substack{0\leqq u\leqq t\\ |z'-y'|\geqq\delta}} q(u,z',y')\cdot P_{x'}[\tau\leqq t]\,.$$

From (6.3) we have

$$\sup_{\substack{0\leqq u\leqq t\\ |z'-y'|\geqq\delta}} q(u,z',y') \leqq M < \infty\,.$$

Hence Theorem 6.3 yields

$$\limsup_{\substack{t\to 0\\ x'\to x\\ y'\to y}} 2t\log q(t,x',y') \leqq \limsup_{\substack{t\to 0\\ x'\to x}} t\log P_{x'}[\tau\leqq t]$$
$$\leqq -\inf_{z\in B}\tfrac{1}{2}d^2(x,z)\,.$$

Letting $\delta\to 0$, we find

$$\limsup_{\substack{t\to 0\\ x'\to x\\ y'\to y}} 2t\log q(t,x',y') \leqq -\tfrac{1}{2}d^2(x,y)\,.$$

LOWER BOUND. Equation (6.4) gives, for every $\Delta > 0$,

(6.5) $$\operatorname*{Inf}_{0\leqq t\leqq t_0}\ \inf_{|x-y|\leqq\Delta t} q(t,x,y) = \theta > 0 \qquad \text{for} \quad t_0 \quad \text{small}\,.$$

Let x, y be points in R_k. Let $x'\to x$ and $y'\to y$. We define the sets

$$U_t = \{z : |z-y'|\leqq t\}\,,$$

and

$$G_t = \{\omega : \omega(t) \in U_t\} .$$

Then,

$$\int_{G_t} p(t, x', v)\, dv = \int_{|z-y'| \leqq t} p(t, x', v)\, dv$$

$$\geqq ct^k \inf_{v \in G_t} p(t, x', v) .$$

Consequently, from Theorem 2.5, we obtain

$$\liminf_{\substack{t \to 0 \\ x' \to x \\ y' \to y}} t \log P_{x'}[G_t] \geqq -\tfrac{1}{2} d^2(x, y) .$$

Rewriting this inequality by introducing the measures P_x^ε, we have

$$\liminf_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log P_{x'}^\varepsilon[G_\varepsilon] \geqq -\tfrac{1}{2} d^2(x, y) ,$$

where we redefine

$$G_\varepsilon = \{\omega : \omega(1) \in U_\varepsilon\} .$$

From relation (6.1), it follows that

$$\liminf_{\substack{\varepsilon \to 0 \\ x' \to x \\ y' \to y}} \varepsilon \log Q_{x'}^\varepsilon[G_\varepsilon] \geqq -\tfrac{1}{2} d^2(x, y) ,$$

or equivalently that

$$\liminf_{\substack{t \to 0 \\ x' \to x}} t \log \int_{G_t} q(t, x', v)\, dv \geqq -\tfrac{1}{2} d^2(x, y) . \tag{6.6}$$

Now,

$$q(t, x', y') = \int q(t(1 - \Delta), x', v) q(\Delta t, v, y')\, dv$$

$$\geqq \int_{G_t} q(t(1 - \Delta), x', v) q(\Delta t, v, y')\, dv$$

$$\geqq \inf_{v \in G_t} q(\Delta t, v, y') \int_{G_t} q(t(1 - \Delta), x', v)\, dv .$$

Using (6.5) and (6.6) we arrive at

$$\liminf_{\substack{t \to 0 \\ x' \to x \\ y' \to y}} t \log q(t, x', y') \geqq -\frac{1}{2(1 - \Delta)} d^2(x, y) .$$

Allowing $\Delta \to 0$ we prove the theorem.

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